Notes

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1 Orders

Definition 1.
A Strict partial order is a structure $\langle X, P\rangle$*, with* P *a binary relation on* X *that is irreflexive (IR) and Transitive (TR):*
(IR) $\forall x : \neg P(x, x)$.
(TR) $\forall x, y, v, w : (P(x, y) \wedge P(y, z)) \rightarrow P(x, z)$.

Alternatively, it is *asymmetric* and *transitive*.

Definition 2.
An Interval order is a structure $\langle X, P\rangle$*, with* P *a binary relation on* X *that is irreflexive and satisfies the Interval order condition (IO):*
(IR) $\forall x : \neg P(x, x)$.
(IO) $\forall x, y, v, w : (P(x, y) \wedge P(v, w)) \rightarrow (P(x, w) \vee P(v, y))$.

Definition 3.
A Semi order is a structure $\langle X, P\rangle$*, with* P *a binary relation on* X *that is irreflexive, and satisfies the Interval order condition (IO) and Semi-transitivity (STr):*
(IR) $\forall x : \neg P(x, x)$.
(IO) $\forall x, y, v, w : (P(x, y) \wedge P(v, w)) \rightarrow (P(x, w) \vee P(v, y))$.
(STr) $\forall x, y, z, v : (P(x, y) \wedge P(y, z)) \rightarrow (P(x, v) \vee P(v, z))$.

Definition 4.
A Weak order is a structure $\langle X, P\rangle$*, with* P *a binary relation on* X *that is irrreflexive (IR), transitive (TR), and almost connected (AC):*
(IR) $\forall x : \neg P(x, x)$.
(TR) $\forall x, y, z : (P(x, y) \wedge P(y, z)) \rightarrow P(x, z)$.
(AC) $\forall x, y, z : P(x, y) \rightarrow (P(x, z) \vee P(z, y))$.

Definition 5.
A Linear order is a structure $\langle X, P\rangle$*, with* P *a binary relation on* X *that is irreflexive (IR), Transitive (TR), and Connected:*

(IR) $\forall x : \neg P(x,x)$.
(TR) $\forall x,y,z : (P(x,y) \wedge P(y,z)) \rightarrow P(x,z)$.
(Con) $\forall x,y,: P(x,y) \vee P(y,z) \vee x = y$.

Let us now define the *indifference relation*, 'I', as follows: $I(x,y)$ iff$_{def}$ neither $P(x,y)$ nor $P(y,x)$. We can also define the following relation: $\approx$ defined as follows: $x \approx y$ iff $\forall z : I(x,z)$ iff $I(y,z)$. If P is a linear order, $I(x,y)$ iff $x = y$. If P is a weak order, I and $\approx$ give rise to the same equivalence relation. If P is a semi-order, I is reflexive and symmetric, but not transitive, but $\approx$ gives rise to an equivalence relation. The same holds if P is an interval order. If P is a strict partial order, there might be two reasons why $I(x,y)$: either they are indifferent, or they are unconnected. In case $\langle X, P\rangle$ is an interval order or more (semi, weak, linear), it always holds that $\forall x,y \in X : P(x,y) \vee P(y,x) \vee I(x,y)$. This makes it possible to represent such orders in terms of numbers (by measurement theory).

Define the relation IP as follows: $xIPy$ iff $\exists z : xIzPy$. Intuitively, x begins earlier than y
Define the relation PI as follows: $xIPy$ iff $\exists z : xPzIy$. Intuitively, x ends earlier than y
If P is an interval-order (or stronger), one can show that IP and PI are weak orders.

Saturation: assume: $\forall x,y \in I : x \not\approx y \rightarrow \exists z : xIzPy \vee xPzIy$.
In case we assume saturation, we can intuitively assume that we start with semi-orders (or interval orders), but define weak orders (the comparative) in terms of IP.

2 Choice functions

2.1 Standards

A much discussed topic in the theory of choice is how a preference order among options can be derived on the assumption that the notion of *choice* is primitive. Assuming a choice function that selects an element from each finite set of options, one can easily show how we can generate a linear order by putting constraints on how this function should behave on different sets of options. Let us define a *choice structure* to be a triple $\langle X, O, C\rangle$, where X is a non-empty set, the set O consists of all finite subsets of A, and the choice function C assigns to each finite set of options $o \in O$ an element of o, $C(o)$, satisfying the following condition:

(LIN) $\forall o, o' \in O : \text{If } (C(o) \in o' \text{ and } C(o') \in o), \text{then } C(o) = C(o')$.

If we say that $x > y$, iff$_{def}$ $C(\{x,y\}) = x$, one can easily show that the ordering as defined above gives rise to a *linear order*

Arrow (1959) already showed how we can generate a weak ordering by putting other constraints. In this case, the choice function C assigns to each finite set of options $o \in O$ a *subset* of o, $C(o)$. Arrow (1959) stated the following principle of choice (C), and the constraints (A1) and (A2) to assure that the choice function behaves in a 'consistent' way:

(C) $\forall o \in O : C(o) \neq \emptyset$.
(A1) If $o \subseteq o'$, then $o \cap C(o') \subseteq C(o)$.
(A2) If $o \subseteq o'$ and $o \cap C(o') \neq \emptyset$, then $C(o) \subseteq C(o')$.

If we say that $x > y$, iff$_{def}$ $x \in C(\{x,y\}) \wedge y \notin C(\{x,y\})$, one can easily show that the ordering as defined above gives rise to a *weak order*.

Condition (A1) is better known as Sen's *Property* α and sometimes called *Chernoff* or *Heritage*. For finitely additive and subtractive domains, it is equivalent to (A1') $C(o \cup o') \subseteq C(o) \cup C(o')$. Condition (A2) is also known as Sen's Property β^+, and is also called *Dual Chernoff*. Taken together with "(EMPTY) If $o \subseteq o'$ and $C(o') = \emptyset$, then $C(o) = \emptyset$" (also assumed by Lewis, and which follows from (C)), it implies both (II) and (III) discussed below. Arrow formulated the choice function as the combination of (A1) and (A2), and called it the axiom of independence of irrelevant alternatives:

(A) If $o \subseteq o'$ and $o \cap C(o') \neq \emptyset$, then $C(o') \cap o = C(o)$.

While condition (A1) expresses some kind of 'contraction consistency' in proceding from larger menus to smaller ones, the following condition proceeds from smaller menus to larger ones:

(II) $C(o) \cap C(o') \subseteq C(o \cup o')$

This axiom is a finitary version of Sen's Property γ which is also called the *Expansion axiom* or *Concordance*:

(II)$^\infty$ $\bigcap\{C(o_i) : i \in I\} \subseteq C(\bigcup\{o_i : i \in I\})$

The following axiom is known as *Aizerman's axiom*:

(III) If $o \subseteq o'$ and $C(o') \subseteq o$, then $C(o) \subseteq C(o')$

If the subset relation in its consequent is replaced by identity, we get what is called *Independence of rejecting the outcast variants* in Aizerman and Malishevski (1981) and Nash's axiom in Suzumura (1983). It is stronger than the *Superset axiom*:

$$\text{If } o \subseteq o' \text{ and } C(o') \subseteq C(o), \text{ then } C(o) \subseteq C(o')$$

Taken together with (A1), the superset axiom implies (III). Condition (III) is independent of condition (II), even for finite X and in the presence of condition (A1).

Suppose that O consists of all finite subsets of I. We can state the following facts:

- If C satisfies EMPTY and (A1), then $<$ is acyclic. (A binary relation $<$ over X is called *n-acyclic*, if no objects $x_1, x_2, ...x_n$ in X form a cycle under $<$, i.e. if a chain $x_1 < x_2 < ... < x_n < x_1$ does not occur. 1-acyclicity is irreflexivity, 2-acyclicity is asymmetry. Notice that acyclycity follows if $<$ is irreflexive (or asymmetric) and transitive.)
- If C satisfies EMPTY and (A1) and (III), then $<$ is transitive.
- If C satisfies EMPTY and (A2), then $<$ is almost connected. (or modular, or negatively transitive, virtually connected, ranked).
  (almost connectedness and asymmetry entail transitivity)
- If C is closed under arbitrary union and satisfies (EMPTY) and (A2), then $<$ is almost connected.

2.2 Characterizing semi-orders

Define $\overline{T}(c)$ to be the complement of $T(c)$, i.e. $\overline{T(c)} = c - T(c)$. Now we can put the following constraints on context structures:

(I) $\forall c \in C : T(c) \neq \emptyset$.
(II) If $c \subseteq \overline{T(c')}$, then $T(c' - c) = T(c')$.
(III) If $T(c) \cap \overline{T(c')} \neq \emptyset$, then $\overline{T(c)} \cap T(c') = \emptyset$.
(IV) If $c \subseteq \overline{T(c')}$ and $T(c') \cap c'' \neq \emptyset$, then $\overline{T(c)} \cap T(c'') = \emptyset$.
(V) If $c \subseteq T(c')$, then $T(c' - c) = T(c') - c$.

Define now the following relations: $xT^{>}y$ iff $\exists c[x, y \in c \wedge x \in T(c) \wedge y \in \overline{T(c)}]$, and $xT^{\geq}y$ iff $\exists c[x, y \in c \wedge x \in T(c)]$. Now one can prove the following facts (i) (Jamison and Lau, 1973): $P^{\geq}$ is a weak order iff constraints (I), (II) and (V) are satisfies; (ii) (Fishburn, 1975): $P^{>}$ is an interval-order iff constraints (I), (II), and (II) are satisfied; (Fishburn, 1975): $P^{>}$ is a semi-order iff constraints (I), (II), (III), and (IV) are satisfied.

2.2.1 Satisficing choice functions

We would like to derive the meaning of 'better than' in terms of the meaning of 'best' – as is assumed if agents are taken to be utility maximizers –, but rather to derive the meaning of 'better than' in terms of the context-dependent meaning of 'good'.[1] What is crucial for the interpretation of the results of our paper is that although 'good' seems to obey axiom (A2), axiom (A1) seems much too strong: (A1) demands that if both x and y are considered to be good in the context of $\{x, y, z\}$, both should considered to be good in the context $\{x, y\}$ as well. But that is exactly what we don't want for a context dependent notion of 'good': in the latter context, we want it to be possible that only x, or only y, is considered to be good. We should conclude that if we want to characterize the behavior of 'good', we should give up on (A1). Unfortunately, by just constraints (C) and (A2) we cannot guarantee that the comparative relation 'better than' behaves as desired. In particular, we cannot guarantee that it behaves almost connected.

To assure that the comparative behaves as desired, we add to (C) and (A2) the Upward Difference-constraint (UD), proposed by Van Benthem (1982). To state this constraint, we define the notion of a difference pair: $\langle x, y\rangle \in D(o)$ iff$_{def}$ $x \in C(o)$ and $y \in (o - C(o))$. Now we can define the constraint:

(UD) $o \subseteq o'$ and $D(o') = \emptyset$, then $D(o) = \emptyset$.

In fact, van Benthem (1982) states the following constraints: No Reversal (NR), Upward Difference (UD), and Downward Difference (DD) (where o^2 abbreviates $o \times o$, and $D^{-1}(o) =_{def} \{\langle y, x\rangle : \langle x, y\rangle \in D(o)\}$):

(NR) $\forall o, o' \in O : D(o) \cap D^{-1}(o') = \emptyset$.
(UD) $o \subseteq o'$ and $D(o') = \emptyset$, then $D(o) = \emptyset$.
(DD) $o \subseteq o'$ and $D(o) = \emptyset$, then $D(o') \cap o^2 = \emptyset$.

[1]Interestingly enough, this is exactly analogue to what Klein (1980) intended to do in linguistics: the meaning of 'taller than' (or 'better than') should be defined in terms of the meaning of 'tall' (or 'good'), not that of 'tallest' (or 'best').

One can show that if constraints (NR), (UD) and (UD) are satisfied, the relations 'I' and 'P' as defined before still have the same properties as before: 'I' is still predicted to be an equivalence relation, while the relation 'P' is still predicted to be (i) irreflexive, (ii) transitive, and (iii) almost connected.

3 Measurement

Krantz et al. 'When measuring some attribute of a class of objects or events ... we associate numbers with the objects in such a way that the properties of the attribute are faithfully represented as numerical properties' (1971, p. 1).

3.0.2 What is measurement?

Measurement of some attribute of a set of things is the process of assigning numbers or other symbols to the things in such a way that relationships of the numbers or symbols reflect relationships of the attributes of the things being measured. A particular way of assigning numbers or symbols to measure something is called a scale of measurement.

Suppose we have a collection of straight sticks of various sizes and we assign a number to each stick by measuring its length using a ruler. If the number assigned to one stick is greater than the number assigned to another stick, we can conclude that the first stick is longer than the second. Thus a relationship among the numbers (greater than) corresponds to a relationship among the sticks (longer than). If we lay two sticks end-to-end in a straight line and measure their combined length, then the number we assign to the concatenated sticks will equal the sum of the numbers assigned to the individual sticks (within measurement error). Thus another relationship among the numbers (addition) corresponds to a relationship among the sticks (concatenation). These relationships among the sticks must be empirically verified for the measurements to be valid.

When an attribute P is suitable for measurement (e.g. temperature, weight), there will be a relation $\geq_P$, where $a \geq_P b$ is true iff a's quantity of P is at least as great as b's. We can formulate some non-numerical principle which govern that relation, and given an appropriate set of principles a *representation theorem* will be provable guaranteeing that numbers can be used to measure the attribute P. More specifically, consider the relational structure given by the ordered pair $\langle X, |ge_P\rangle$, where X is the set of objects or events that have P. A representation theorem states that there is a homomorfism, ϕ, from this relational structure into $\langle \mathbf{R}, \geq\rangle$ such that

$$(R) \quad a \geq_P b \text{ iff } \phi(a) \geq \phi(b)$$

Take the measure of temperature and the relation *at least as hot as*, $\geq_H$, holding between pairs of objects. Using ϕ, numbers are assigned to objects according to their temperature: $\phi(a) \geq \phi(b)$ then reflects the fact that a is at least as hot as b, i.e. $a \geq_H b$, so $\phi(a) = \phi(b)$ is true iff a and b have the same temperature. Certain axioms are needed to prove any representation theorem. They typically include connectedness and transitivity.

In general, the homomorfism or scale ϕ will not be unique (consider e.g. the use of different scales for measuring temperature). But a *uniqueness theorem* will be provable which states that it is unique *up to a certain type of transformation.* Such a theorem thus characterizes the permissible transformations of any legitimate scale, and its breadth will depend on the principles governing $\geq_P$. Features of the numerical structure which correspond to genuine features of the attribute will be shared by all acceptable numerical assignments, and different uniqueness theorems lead to different answers to questions such as 'if two numbers add together to equal a third, does this correspond to a genuine relation between the objects assigned those numbers?'

Different uniqueness theorems determine different types of measurement scale.

3.0.3 What are levels of measurement?

There are different levels of measurement that involve different properties (relations and operations) of the numbers or symbols that constitute the measurements. Associated with each level of measurement is a set of permissible transformations. Permissible transformations are transformations of a scale of measurement that preserve the relevant relationships of the measurement process. In the example of measuring sticks, changing the unit of measurement (say, from centimeters to inches) multiplies the measurements by a constant factor. This multiplication does not alter the correspondence of the relationships 'greater than' and 'longer than', nor the correspondence of addition and concatenation. Hence, change of units is a permissible transformation with respect to these relationships.

Nominal:

- Two things are assigned the same symbol if they have the same value of the attribute.
- Permissible transformations are any one-to-one or many-to-one transformation, although a many-to-one transformation loses information.
- Examples: numbering of football players; numbers assigned to religions in alphabetical order, e.g. atheist=1, Buddhist=2, Christian=3, etc.

Ordinal:

- Things are assigned numbers such that the order of the numbers reflects an order relation defined on the attribute. Two things x and y are assigned numbers $\phi(x)$ and $\phi(y)$ such that if $\phi(x) > \phi(y)$, then $x > y$.
- Permissible transformations are any order-preserving transformation. In other words, we can allow a transformation $f(x)$ of the values assigned by ϕ whenever f is a strictly increasing function (i.e., one for which $f(x) < f(y)$ whenever $x < y$). For instance, a transformation that squares the original values will be acceptable.
- Examples: Moh's scale for hardness of minerals; grades for academic performance (A, B, C, ...); blood sedimentation rate as a measure of intensity of pathology.

Interval (or *Intensional magnitudes*):

- Things are assigned numbers such that differences between the numbers reflect differences of the attribute. If $\phi(x) - \phi(y) > \phi(u) - \phi(v)$, then $x - y > u - v$. Thus, if the invervals between two pairs of values are the same, they must remain the same under any permissible transformation, and other ratios between intervals are similarly invariant.
- Permissible transformations are any affine transformation $f(x) = \alpha x + \beta$, where α and β are constants; another way of saying this is that the *origin*/zero (due to β) and *unit* (due to α) of measurement are *arbitrary.* (note though, 0 and 1 play an important part)
- Examples: temperature in degrees Fahrenheit or Celsius; calendar date. Note: the choice of zero is arbitrary

Log-interval:

- Things are assigned numbers such that ratios between the numbers reflect ratios of the attribute. If $\frac{m(x)}{m(y)} > \frac{m(u)}{m(v)}$, then $\frac{a(x)}{a(y)} > \frac{a(u)}{a(v)}$.
- Permissible transformations are any power transformation $f(x) = \alpha \times x^{\beta}$, where α and β are constants.
- Examples: density (mass/volume); fuel efficiency in mpg.

Ratio (or *extensional* or *additive* magnitudes):

- Things are assigned numbers such that not only orderings and differences, but also ratios between the numbers reflect differences and ratios of the attribute.

- Permissible transformations are any linear (similarity) transformation $f(x) = \alpha x$, where α is a constant; another way of saying this is that the *unit* of measurement is *arbitrary* (but not the zero). Note: choice of *unit* is generally still *arbitrary* (consider scales employing meters and feet). Sums of values are also significant here: suppose $\phi(x) + \phi(y) = \phi(z)$, then $\alpha(\phi(x)) + \alpha(\phi(y)) = \alpha(\phi(z))$, for any α, thus $\phi(x) + \phi(y) = \phi(z)$ represents a genuine relation between x, y and z themselves.

- Examples: Length in centimeters; duration in seconds; temperature in degrees Kelvin.

Notice that the symbol "$\circ$" indicates a specified procedure for joining a and b. The rule of additivity holds: $M(a \circ b) = M(a) + M(b)$ Weight is additive: it does not matter exactly how the two bodies are placed together on the scale, it *does* matter for the case of length. However, we call length an additive magnitude because there is an operation of joining to provide a basis for the additive principle. But not all magnitudes are additive. Examples are *temperature*, *pitch of sounds* and the *hardness of bodies*.

Absolute:

- Things are assigned numbers such that all properties of the numbers reflect analogous properties of the attribute.

- The only permissible transformation is the identity transformation.

- Examples: number of children in a family, probability.

These measurement levels form a partial order based on the sets of permissible transformations:

3.1 Numbers or intervals?

It is well-known that in case 'P' gives rise to a weak order, it can be represented numerically by a real valued function u such that for all $x, y \in I$: xPy iff $u(x) > u(y)$, and xIy iff $u(x) = u(y)$.
In case P is a semi-order, xPy iff $u(x) > u(y) + \epsilon$ and $|u(x) - u(y)| < \epsilon$. Intuitively, this means that x and y are modeled by intervals, such that all intervals are equally long
In case P is an interval order xPy iff $u(x) - u(y) > \epsilon(y)$. Intuitively, this means that all elements of I are modeled by intervals, but that they don't have to be equally long.

4 Comparatives: A degree approach

The general argument in favor of degrees is formed by sentences that either explicitly talk about degrees ((1-a), (1-b), (1-c)), or by so-called subdeletion cases ((2-a) and (2-b)):

(1) a. John is 5 foot tall.
    b. John is 2 centimeter taller than Mary.
    c. John is twice as tall as Mary.

(2) a. John is taller than he is wide.
    b. John is taller than Mary is wide.

4.1 Data: Forms of comparison

(3) a. The Empire State Building is taller than the Chrisler Building (inequality)
    b. WTC is less tall than the Sears Tower (negative inequality)
    c. Bertha is as tall as Martha (equality)

(4) a. *The degree to which* the Empire State Building is tall is *greater than the degree to which* the Chrysler Building is tall.
    b. *The degree to which* the WTC is tall *is smaller than the degree to which* the Sears Tower is tall.
    c. *The degree to which* Bertha is tall *is at least as great as the degree to which* Martha is tall.

(5) Multi head constructions (subdeletion)
    a. More cats ate mice than dogs rats.
    b. the number of cats that ate mice is greater than the number of dogs that ate rats AND the number of mice that were eaten by cats is greater than the number of rats that are eaten by cats.

(6) Measure phrases and differentials
    a. The Woolworth Building is 767 feet tall.
    b. The Empire State building is 2 feet taller than the Chrysler Building.
    c. The Sears tower is twice as tall as the Woolworth Building.

(7) a. *The degree to which* the Woolworth Building is tall *equals* 767 feet.
    b. *The degree to which* the Empire State Building is tall *equals the degree to which* the Chrysler Building is tall *plus* 2 feet.
    c. *The degree to which* the Sears tower is tall *equals the degree to which* the WOolworth Building is tall *times 2*.

4.2 Measurement

Prediction: Measure phrases and differentials involve addition and multiplication. So we expect to get them only if the property in question is associated with an order and a concatenation operation.

(8) a. Peter bought twenty liters of gold. (volume)
b. *Peter bought twenty degrees of gold. (temperature)

4.3 Distribution of amount terms

+ Occur with 'measure' adjectives in the positive: *deep, wide, long, old, tall, high, thick* in English, i.e. age and distance in various dimensions: length/breath/depth/height.

(9) a. John is 5 years old. (additive)
b. The ice was 5 cm thick (additive)

(10) a. *The water was 75 degrees warm. (non-additive)
b. *The driver drove 15 mph fast. (non-additive)

(11) a. *The concert was only 40 minutes long. (additive)
b. *The suitcase is 20 kilos heavy (additive)
c. *The apartment is 1000 square feet big (additive)

+ There are cross linguistic differences. In German the counterparts of (11) are acceptable. But, in this language we may have adjectives that associate non-additive structure.

(12) a. Das Konzert war nur 40 Minuten lang.
b. Der Koffer ist 20 Kilo schwer.
c. Die Wohnung ist 90 Quadratmeter gross.

(13) Das Wasser war 75 Grad warm. (non-additive)

+ Occur with all size-denoting and some value adjectives in the comparative and the equative: we distinguish additive measures as in (14-a)-(14-c) and multiplicative measures as in (14-d)-(14-e)

(14) a. The brown pencil is 2 cm longer than the blue pencil. (length)
b. Peter invited two more people than he ordered pizza (cardinality)
c. John drove 2 miles an hour faster than Bill. (speed)
d. the water is two degrees warmer today than yesterday. (hotness)
e. In Florida it is today twice as warm as in Colorado.

(15) a. The diamond is twice as big as the ring. (mass)
b. Little Snow-White is thousand times fairer than you. (beauty)

$\Rightarrow$ It seems that a naive mathematics of measurement theory, the concept of additivity in particular, does not help to predict the distribution of amount terms in adjectival natural language comparison.

4.3.1 Semantics

Assume a set of degrees

(16) a. $\llbracket tall \rrbracket = \lambda w \lambda d \lambda x.x$ is tall to degree d in w
b. $\llbracket fast \rrbracket = \lambda w \lambda d \lambda x.x$ is fast to degree d in w
c. $\llbracket beautiful \rrbracket = \lambda w \lambda d \lambda x.x$ is beautiful to degree d in w

(17) a. $\llbracket POS \rrbracket = \lambda D.\exists d[d \geq s \; \& \; \iota d'.D(d') = d]$
b. $\llbracket \textit{at least } 767 \textit{ feet} \rrbracket = \lambda D.\exists d[d \geq 767 \text{ feet} \; \& \; D(d)]$
c. $\llbracket \textit{at least} \rrbracket = \lambda d \lambda N \exists n[n \geq n \; \& \; N(n)]$
d. $\llbracket 767 \rrbracket =$ the number 767.
e. $\llbracket \textit{the foot} \rrbracket =$ the standard unit with the name "foot".

(18) $\llbracket POS \rrbracket = \lambda u \lambda n \lambda D.\exists \phi[\phi(u) = 1 \; \& \; \phi(\iota d.D(d) = n]]$

(19) $\llbracket -er \rrbracket = \lambda D_2 \lambda D_1.\iota d.D_1(d) > D_2(d)$.

(20) $\llbracket -er \rrbracket = \lambda u \lambda n \lambda D_2 \lambda D_1.\exists \phi[\phi(u) = 1 \; \& \; \phi(\iota d.D_1(d))\phi(\iota d.D_2(d)) + n]$

(21) $\llbracket less \rrbracket = \lambda u \lambda n \lambda D_2 \lambda D_1.\exists \phi[\phi(u) = 1 \; \& \; \phi(\iota d.D_1(d))\phi(\iota d.D_2(d)) - n]$

4.4 Problems

- We cannot predict that amount terms do not combine with negative polar adjectives.

  (22) ?The Woolworth Building is 767 feet short.

- Comparatives and modal operators.

  (23) a. A polar beer could be bigger than a grizzly bear could be.
  b. The Sears Tower was taller than any other building.

  (24) a. A possible polar bear size is greater than the highest possible size of a grizzly bear.
  b. The size the Sears tower was taller than the tallest of all the other buildings.

(25) The degree d such that it is possible that a polar bear is d-tall is greater than the degree such that it is possible that grizzly bear is d-tall.

⇒ introduction of the maximality operator.

(26)
a. $[\![-er]\!] = \lambda D_2 \lambda D_1.\exists\phi[\phi(MAX[D_1]) \geq \phi(MAX[D_2])]$
b. $MAX(P) = \iota d.\forall d'[P(d') \rightarrow d' \leq d]$
c. $\exists\phi[\phi(MAX[D_1]) \geq \phi(MAX[D_2]]$, where
d. $D_1 = \lambda d.\exists w[w \in Acc(w_0) \wedge \exists x[x$ is a polar bear in w & has size x in $w]]$
e. $D_2 = \lambda d.\exists w[w \in Acc(w_0) \wedge \exists x[x$ is a grizzly bear in w & has size x in $w]]$

5 An extent account: Comparison and polar opposition

5.1 The data

The difference between positive polar and negative polar adjectives

(27)
a. ?Carmen is taller than Alice is short. (cross polar anomaly)
b. ?Alice is 4 feet short. (measure phrases)

(28) Carmen is taller than Alice ⇔ Alice is shorter than Carmen. **Claim**: the degree approach cannot explain cross polar anomalies.

Assumptions:

- Gradable adjectives assign degrees to objects (vs. relational theories)
- Degrees are equivalent classes (just sets of objects)
- The degree to which Carmen is tall = the degree to which Carmen is short.
- The morphologically marked adjective determines the empirical ordering relation between the degrees.

⇒ **Problematic**: This a rather naive degree account. In Cresswell's (1979) analysis it does not go through. It is not clear which compositional method lead to the paraphrases.

(29) a. Carmen is taller than Alice is (tall).

b. The degree to which Carmen is tall $>_{tall}$ the degree to which Alice is tall.

(30) a. Carmen is taller than Alice is short.
b. the degree to which Carmen is tall $>_{tall}$ the degree to which Alice is short.

5.2 Kenedy's semantics

(31) The values compared are intervals (sets of points d on a scale S).
$\forall p_1 p_2 p_3 [p_1 p_2 \in d \wedge p_3 \in S \wedge p_1 > p_3 > p_2 \rightarrow p_3 \in d]$

(32) a. Positive and negative extents/degrees
b. $POS(S) = \{d \subseteq S | \exists p_1 [p_1 \in d \wedge \forall p_2 \in S [p_2 \geq p_1 \rightarrow p_2 \in d]]$
c. $NEG(S) = \{d \subseteq S | \exists p_1 [p_1 \in d \wedge \forall p_2 \in S [p_1 \geq p_2 \rightarrow p_2 \in d]]$

(33) a. Positive and negative projection of x on a scale S
b. $pos_{length}(a) = \{p' \in S | p' \leq$ the point p s.t. a has p on the scale associated with length$\}$
c. $neg_{length}(a) = \{p' \in S | p'$ the point p s.t. a has p on the scale associated with length$\}$

5.2.1 Positive and negative polar adjectives

(34) a. $[\![tall]\!] = \lambda w \lambda d_D \lambda x_e . pos_{length(w)}(x) \supseteq d$
b. $[\![short]\!] = \lambda w \lambda d_D \lambda x_e . neg_{length(w)}(x) \supseteq d$

5.2.2 Measure phrases

(35) $[\![sixfeet]\!] = pos_{length}(6feet)$

(36) a. Bertha is six feet tall.
b. $[\![tall(w_0)(sixfeet)(Bertha)]\!]^g = 1$ iff $pos_{length(w_0)}(Bertha) \geq pos_{length}(6feet)$

(37) a. Bertha is six feet short.
b. $[\![short(w_0)(sixfeet)(Bertha)]\!]^g = 1$ iff $neg_{length(w_0)}(Bertha) \geq pos_{length}(6feet)$

6 Event structures

Definition 6.
A Russell event structure $\Sigma_R = \langle E, <, \sim, :, <_B, < E \rangle$, *consists of a non-empty set* E *of events together with five binary relations 'before' (<), 'overlap' (*$\sim$*), 'meet' (:), 'begins before' (*$<_B$*), and 'ends-before (*$<_E$*) such that:*

$<$ *is a strict partial oder;*
$\sim$ *is reflexive and symmetric;*
$<$ *and* $\sim$ *are disjoint;*
$< \cup \sim$ *is complete*
$(e < e' \wedge e' \sim e'' \wedge e'' < e''') \rightarrow e < e'''$

The relations ':', $<_B$, and $<_E$ are defined as follows:
$e : e'$ iff$_{def}$ $e < e' \wedge \neg\exists e'', e'''(e'' \sim e''' \wedge e < e''' \wedge e'' < e')$
$e <_B e'$ iff$_{def}$ $\exists e''(e'' \sim e \wedge e'' < e')$
$e <_E e'$ iff$_{def}$ $\exists e''((e < e'' \wedge e'' \sim e')$.

Russell (1926, 1956) offers a way of constructing instants. His motivation is that he wants to show that our conception of abstract instants is derived from the events and the temporal relationships we perceive. Russell defines instants as maximal sets of pairwise overlapping events. he shows that instants thus constructed have the properties we normally require of instants of time, such as that they form a strict linear ordering. So Russell's theory of time is part of his 'Logical Atomicism'.

Formally, Russell's definition of instants can be put as follows (slightly modified from Kamp, 1979):

Definition 7.
Let $\Sigma_R = \langle E, <, \sim, :, <_B, < E\rangle$ *be a Russell event structure. An instant* i *is a subset of* E *such that:*
$\forall e, e' \in i : e \sim e'$
$\forall e \notin i : \exists e' \in i : e \not\sim e'$

We denote the set of instants of Σ_R as $I(\Sigma_R)$.

Definition 8.
(Kamp). Let i *and* i' *be any two instants of* $I(\Sigma_R)$*. Then:* $e < e'$ *iff*$_{def}$ $\exists e \in i : \exists e' \in i : e < e'$*. We call* $\tau(\Sigma_R) = \langle I(\Sigma_R), <\rangle$ *the instant structure derived from* Σ_R*. One can prove that* $\tau(\Sigma_R)$ *is a strict linear ordering.*

Definition 9.
A Thomason (1989) event structure $\Sigma_R = \langle E, <, \sim, :, <_B, < E\rangle$*, consists of a non-empty set* E *of events together with five binary relations 'before' (*$<$*), 'overlap' (*$\sim$*), 'meet' (*$:$*), 'begins before' (*$<_B$*), and 'ends-before (*$<_E$*) such that:*
$<$ *is an interval order;*
$<_B$ *and* $<_E$ *are asymetric and satisfy almost connectedness (transitivity follows)*
$(e < e' \wedge \neg e < e'') \rightarrow e'' <_B e')$

$(e < e' \wedge \neg e'' < e') \rightarrow e <_B e'')$
$e : e' \rightarrow e < e'$
$(e : e' \wedge e < e'')\neg e'' <_B e')$
$(e : e' \wedge e'' < e')\neg e' <_B e'')$
$(\neg e <_E e' \wedge \neg e'' <_E e \wedge \neg e' <_B e''' \wedge \neg e''' <_B e') \rightarrow (e : e' \leftrightarrow e'' : e''')$

The relations '$\sim$' is defined as follows: $e \sim e'$ iff$_{def}$ $\neg(e < e') \wedge \neg(e' < e)$