# Structures for Semantics: last assignment

[You may e-mail your work until June 5 to r.a.m.vanrooij@uva.nl, or hand it in at that date. In case you have any questions about the exercises, please contact me (Robert van Rooij, room 213, phone 525-4551, r.a.m.vanrooij@uva.nl)]

## 1 Fuzzy logic

A $t$-norm $\mathbf{t}$ is a binary function from $[0,1]^2$ to [0,1] that is commutative, associative and monotone increasing with 1 as neutral element and 0 as zero element. That means that for arbitrary $x, y, z, u, v \in [0,1]$ the following holds:

(1) $x\mathbf{t}y = y\mathbf{t}x$

(2) $x\mathbf{t}(y\mathbf{t}z) = (x\mathbf{t}y)\mathbf{t}z$

(3) if $x \leq u$ and $y \leq v$, then $x\mathbf{t}y \leq u\mathbf{t}v$

(4) $x\mathbf{t}1 = x$ and $x\mathbf{t}0 = 0$

Look at two arbitrary fuzzy sets $m_A$ and $m_B$ (where $m_A$ is a function assigning to every element of D a number in $[0,1]$). For any $t$-norm $\mathbf{t}$ one can define the *intersection* $\cap_{\mathbf{t}}$ for vague sets as follows:

$$m_{A\cap_{\mathbf{t}}B}(x) = m_A(x)\mathbf{t}m_B(x), \text{ for all } x \in D$$

(a) Show that $m_{A\cap_{\mathbf{t}}B} \subseteq m_A$, where $m_A \subseteq m_B$ iff$_{def}$ $\forall x \in D : m_A(x) \leq m_B(x)$.
(Hint: notice that from (3) and (4) it follows that $x\mathbf{t}y \leq x\mathbf{t}1 = x$ and $x\mathbf{t}y \leq y$)

(b) Show that there is a $\mathbf{t}$-function satisfying the above constraints such that $m_{A\cap_{\mathbf{t}}A} \neq m_A$.

(c) Show that for all $t$-norms $\mathbf{t}$ it holds that $x\mathbf{t}y \leq min\{x, y\}$.

## 2 Contextual refinement

Let $\mathcal{R}$ and $\mathcal{R}^{\mathcal{D}}$ be the relations 'being visibly shorter than' and 'being indirectly visibly shorter than', respectively, as defined in the handout given in the vagueness-class. Show the following:

1. $\mathcal{R} \subseteq \mathcal{R}^{\mathcal{D}}$;
2. $\mathcal{R}^{\mathcal{D}}$ is irreflexive
3. $\mathcal{R}^{\mathcal{D}}$ is transitive, if $\mathcal{R}$ is.

## 3 Semi-orders and equivalence relations

In class we showed how we can generate a linear order from a weak order $\langle I, > \rangle$. First, we define the relation '$\sim$' as $x \sim y$ iff$_{def}$ $x \not> y$ and $y \not> x$. Then we observed that '$\sim$' is an equivalence relation that gives rise to the following set of equivalence classes: $\{[x]_\sim : x \in I\}$. Then we looked at the structure $\langle \{[x]_\sim : x \in I\}, >^* \rangle$, where '$>^*$' was defined as follows: $X >^* Y$ iff$_{def}$ $\exists x \in X : \exists y \in Y : x > y$. Then we proved that $\langle \{[x]_\sim : x \in I\}, >^* \rangle$ is a linear order.

Let us now do something very similar for going from semi-orders $\langle I, > \rangle$ to weak orders. We know that '$\sim$' as defined as usual – $x \sim y$ iff$_{def}$ $x \not> y$ and $y \not> x$ – does now not (necessarily) gives rise to an equivalence relation. However, we can define the following relation: '$\approx$' as follows: $x \approx y$ iff$_{def}$ $\forall z \in I : x \sim z$ iff $y \sim z$.

(a) Show that '$\approx$' is indeed an equivalence relation, if $\langle I, > \rangle$ is a semi-order.

Now we look at the structure $\langle \{[x]_\approx : x \in I\}, >^* \rangle$, where '$>^*$' is defined as in the previous case: $X >^* Y$ iff$_{def}$ $\exists x \in X : \exists y \in Y : x > y$.

(b) Does it now hold that $\langle \{[x]_\approx : x \in I\}, >^* \rangle$ is a weak order? If so, show me. If not, give a counterexample.

## 4 Intervals and witnesses

Take an arbitrary interval structure $\Sigma_R = \langle I, < \rangle$, where $I$ is a set of intervals and $<$ satisfies the conditions for interval orders. Define $x \sqsubseteq y$ iff$_{def}$ $\forall z[y < z \rightarrow x < z] \wedge \forall z[z < y \rightarrow z < x]$. It is easy to prove that '$\sqsubseteq$' is reflexive, transitive, and antisymmetric, and thus is a partial order. In terms of '$<$' and '$\sqsubseteq$' we can now define three new principles, Convexity, Monotonicity, and Conjunction (the relation '$\sim$' is defined as usual):

(CONV) $x < y < z \rightarrow \forall u[x \sqsubseteq u \wedge z \sqsubseteq u \rightarrow y \sqsubseteq u]$
(MON) $x < y \rightarrow \forall z[z \sqsubseteq x \rightarrow z < y]$
(CONJ) $x \sim y \rightarrow \exists z \sqsubseteq x[z \sqsubseteq y \wedge \forall u \sqsubseteq x[u \sqsubseteq y \rightarrow u \sqsubseteq z]]$

(a) In class we showed that (CONV) holds in every interval order. What about (MON) and (CONJ)? If they hold, show me, if not, give a counterexample.

Now define the relations 'begins before', $<_B$, 'ends before', '$<_E$', 'begins at the same time', '$=_B$', and 'ends at the same time, '$=_E$' as follows: (i) $x <_B y$ iff$_{def}$ $\exists z[x \sim z \wedge z < y]$, (ii) $x <_E y$ iff$_{def}$ $\exists z[x < z \wedge z \sim y]$, (iii) $x =_B y$ iff$_{def}$ $x \not<_B y$ and $y \not<_B x$, and (iv) $x =_E y$ iff$_{def}$ $x \not<_E y$ and $y \not<_E x$. If we now define the relation '$\sqsubset$' in the expected way ($x \sqsubset y$ iff$_{def}$ $x \sqsubseteq y \wedge y \not\sqsubseteq x$]), we can formulate the following constraint:

(DIFF) $(x \sqsubset y \rightarrow x =_E y]) \rightarrow \exists z[z \sqsubset y \wedge z =_B y \wedge z \not\sim x]$

(b) Give me an interval structure where (DIFF) does not hold.