Basic Generalized Quantifier Theory
Handout

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Additional literature: Peters and Westerståhl (2006)

1 Aristotelian syllogistic

(1) All Greeks are sailors.

(2) Some Sailors are scared.

(3) No Greeks are scared.

(4) Not all Greeks are immortal.

2 FO (un)definability

Peirce and Frege introduced $\forall$ and $\exists$.
Tarski equipped FO with model-theoretic semantics.

$$M \models \forall x \varphi(x) \iff \text{for all } a \in M, M \models \varphi(a).$$

$$M \models \exists x \varphi(x) \iff \text{for some } a \in M, M \models \varphi(a).$$

We can define in FO all Aristotelian quantifiers and more:

(5) At least 3 people are interested in GQs.

(6) At most 2 students like Montague grammar.

(7) Exactly 4 researchers believe that $P = NP$.

But FO is not expressive enough to account for all instances of quantification in NL, e.g.:

(8) There are infinitely many prime numbers.

(9) There is an even number of students in this room.

(10) Most students are already bored with GQT.

Proposition 1 *Finiteness, infinity, parity and* Most *are not FO-definable.*

3 Second-order logic

Most As are $B \iff \exists f : A \longrightarrow B$ such that f is surjective but not injective.

Homework 1 *Define in SO the following quantifiers:* There are infinitely many As, There is an even number of As.

4 Generalized Quantifiers

Definition 1 **((Mostowski, 1957; Lindström, 1966))** *A generalized quantifier* Q *of type* $(n_1, \ldots, n_k)$ *is a class of structures of the form* $M = (U, R_1, \ldots, R_k)$*, where* R_i *is a subset of* U^{n_i}[1].

Definition 2 *A generalized quantifier* Q *of type* $(n_1, \ldots, n_k)$ *is a functor assigning to every set* M *a* k*-ary relation* $\mathsf{Q}(M)$ *between relations on* M *such that if* $(R_1, \ldots, R_k) \in \mathsf{Q}(M)$ *then* R_i *is an* n_i*-ary relation on* M*, for* $i = 1, \ldots, k$.

In other words, Q is a functional relation associating with each model M a relation between relations on M. Hence,

[1]Originally we also demand that Q is closed under isomorphism, but we will consider this condition separately later.

If we fix a model M we have the following equivalence:

$$(U, {R_1}^M, \ldots, {R_k}^M) \in \mathsf{Q} \iff \mathsf{Q}_M R_1 \ldots R_k, \text{where } {R_i}^M \subseteq U^{n_i}.$$

As an example consider the quantifier Most of type (1, 1). It corresponds to the following class of models:

$$\mathsf{Most} = \{(U, A^M, B^M) : card(A^M \cap B^M) > card(A^M - B^M)\}.$$

In a model M the statement $\mathsf{Most}_M AB$ says that $card(A^M \cap B^M) > card(A^M - B^M)$.

Syntactically a quantifier Q of type $(n_1, \ldots, n_k)$ binds $m = n_1 + \ldots + n_k$ first-order variables, and k formulae. If for all i: $n_i \leq 1$, then we say that quantifier is *monadic*, otherwise we call it *polyadic*.

By $FO(\mathsf{Q})$ we mean the elementary logic enriched by the quantifier Q. We define the set of formulae of the logic $FO(\mathsf{Q})$ in the standard way adding the following rule:
if $\psi_1, \ldots, \psi_k$ are formulae and $\bar{x}$ is a sequence of different first-order variables of length m then $\mathsf{Q}\bar{x}(\psi_1, \ldots, \psi_k)$ is a formula.

Let us observe that this definition can be modified according to common notational habits as follows. Q is treated as binding $n = max(n_1, \ldots, n_k)$ variables in k formulae. For example, the quantifier All of type $(1, 1)$ which expresses the property $\forall x(P_1(x) \Rightarrow P_2(x))$ can be written according to the first convention as:

$$\mathsf{All}\ xy\ (P_1(x), P_2(y))$$

and according to the modified one as:

$$\mathsf{All}\ x\ (P_1(x), P_2(x)).$$

The satisfaction definition is extended by the rule for $\mathsf{Q}\bar{x}(\psi_1, \ldots, \psi_k)$ in the following way:

$$M \models \mathsf{Q}\bar{x}(\psi_1, \ldots, \psi_k)[\bar{a}] \iff (\psi_1^{M,\bar{a},\bar{x}}, \ldots, \psi_k^{M,\bar{a},\bar{x}}) \in \mathsf{Q}(|M|),$$

where $|M|$ is the universe of the model M and $\varphi^{M,\bar{a},\bar{x}}$ is the relation defined by formula φ in the model M under valuation $\bar{a}$ relatively to the variables $\bar{x}$, i. e. $\varphi^{M,\bar{a},\bar{x}} = \{(b_1, \ldots, b_k) \in |M|^k : M \models \varphi[\bar{b}]$, where $\bar{b}$ is a valuation assigning b_1 to $x_1, \ldots, b_s$ to x_s and for all other variables the same as $\bar{a}\}$.

5 Examples of GQs

$$\exists = \{(U, A) : A \subseteq U \wedge A \neq \emptyset\}.$$

$$\forall = \{(U, A) : A = U\}.$$

$$\exists^{=m} = \{(U, A) : A \subseteq U \wedge card(A) = m\}.$$

$$\mathsf{D_n} = \{(U, A) : A \subseteq U \wedge card(A) = k \times n\}.$$

$$\mathsf{Most} = \{(U, A, B) : A, B \subseteq U card(A \cap B) > card(A - B)\}.$$

$$\mathsf{Equal} = \{(U, A_1, \ldots, A_n) : A_1, \ldots, A_n \subseteq U card(A_1) = \ldots = card(A_n)\}.$$

Homework 2 *Define as GQs: all Aristotelian quantifiers,* more than*,* only*,* all but John*,* John *as Montagovian individual,* John's*.*

6 Boolean operations on quantifiers

To account for complex NPs: at least 5 or at most 10 departments, between 100 and 200 students, not all residents, ...

Definition 3

$$(\mathsf{Q} \wedge \mathsf{Q}')_M(R_1, \ldots, R_k) \iff \mathsf{Q}_M(R_1, \ldots, R_k) \text{ and } \mathsf{Q}'_M(R_1, \ldots, R_k).$$

$$(\mathsf{Q} \vee \mathsf{Q}')_M(R_1, \ldots, R_k) \iff \mathsf{Q}_M(R_1, \ldots, R_k) \text{ or } \mathsf{Q}'_M(R_1, \ldots, R_k).$$

$$(\neg\mathsf{Q})_M(R_1, \ldots, R_k) \iff \text{not } \mathsf{Q}_M(R_1, \ldots, R_k).$$

(11) It is not the case that most students passed. (outer negation)

(12) Most students did not passed. (inner negation)

Definition 4 *Let* Q *be of type (1, 1):*

$$(\mathsf{Q}\neg)_M(A, B) \iff Q_M(A, M - B).$$

$$Q^d = \neg(Q\neg) = (\neg Q)\neg.$$

Definition 5 *For* Q *of type (1, 1) the square of opposition for* Q *is* $square(\mathsf{Q}) = \{\mathsf{Q}, \neg\mathsf{Q}, \mathsf{Q}\neg, \mathsf{Q}^d\}$.

Hypothesis 1 *NL determines (NPs) denotations are closed on Boolean operations.*

7 Relativization

Definition 6 *Let* Q *be of type* $(n_1, \ldots, n_k)$ *then* Q^{rel} *has the type* $(1, n_1, \ldots, n_k)$ *and is defined for* $A \subseteq M, R_i \subseteq M^{n_i}, 1 \leq i \leq k$ *as follows:*

$$\mathsf{Q}_M^{rel}(A, R_1, \ldots, R_n) \iff \mathsf{Q}_A(R_1 \cap A^{n_1}, \ldots, R_n \cap A^{n_k}).$$

In particular, for Q *of type (1) we have:*

$$\mathsf{Q}_M^{rel}(A, B) \iff \mathsf{Q}_A(A \cap B).$$

8 Some properties of GQs

Barwise and Cooper (1981) started detailed study of NL quantifiers in search for semantic universals (constraints satisfied by all NL determiners) .

8.1 Topic neutrality: isomorphism

Definition 7 *A quantifier* Q *satisfies ISOM iff whenever* M *and* M' *are isomorphic, then* $(M \in Q \iff M' \in Q)$.

Proposition 2 *A type (1) quantifier* Q *satisfies ISOM iff for any universe* M, M'*, and any* $A \subseteq M, A' \subseteq M'$*:*

$$\text{If } card(A) = card(A') \text{ and } card(M - A) = card(M' - A'),$$

$$\text{then } \mathsf{Q}_M(A) \iff \mathsf{Q}_{M'}(A').$$

Definition 8 *If* Q *is an ISOM type (1) quantifier, define, for any cardinal number* k, m*:*

$$\mathsf{Q}(k, m) \iff \text{there are } M, A \subseteq M \text{ such that}$$

$$card(M - A) = k, card(A) = m, \text{ and } \mathsf{Q}_M(A).$$

Homework 3 *Prove that the class of ISOM type (1) quantifiers is closed under Boolean operations, including inner negations and duals.*

8.2 Domain independence: EXT

Definition 9 *A quantifier of type* $(n_1, \ldots, n_k)$ *satisfies EXT iff the following holds:*

If $R_i \subseteq M^{n_i}, 1 \leq i \leq k, M \subseteq M'$, *then* $\mathsf{Q}_M(R_1, \ldots, R_k) \iff \mathsf{Q}_{M'}(R_1, \ldots R_k)$.

Particularly, if Q *is of type (1), then*

$$\mathsf{Q}_M(A) \iff \mathsf{Q}_{M'}(A).$$

Proposition 3 *Let* Q *be a type (1).* Q *is EXT iff*

$$A \subseteq M \wedge A \subseteq M' \text{ then } \mathsf{Q}_M(A) \iff \mathsf{Q}_{M'}(A).$$

Proposition 4 *If* Q *is of type (1) EXT and ISOM, then we can seen it as a class of natural numbers, i.e.:*

$$\text{If } \mathsf{Q}(k, m), \text{ then for all } k' \ \mathsf{Q}(k', m).$$

Homework 4 *Prove that relativized quantifiers, of any type, satisfy EXT.*

8.3 Conservativity

Definition 10 *A type (1, 1) quantifier* Q *is CONS iff for all* M *and all* $A, B \subseteq M$*:*

$$\mathsf{Q}_M(A, B) \iff Q_M(A, A \cap B).$$

(1) All boys walked.

(2) All boys are boys that walked.

9 CE quantifiers and number triangle

Definition 11 *If a quantifier satisfy ISOM, CONS, and EXT, then we call it CE quantifier.*

Hypothesis 2 *NL quantifiers are CE quantifiers.*

Definition 12 *Let* Q *be CE. We define binary relation also denoted by* Q*:*

$$\mathsf{Q}(k,m) \iff \text{there is } M, A \subseteq M \text{ such that}$$

$$card(M-A)=k, card(A)=m, \text{ and } Q_M(A).$$

Proposition 5 *If* Q *is CE, then for all* M *and all* $A,B \subseteq M$ *we have:*

$$\mathsf{Q}_M(A,B) \iff \mathsf{Q}(card(A-B), card(A\cap B)).$$

Homework 5 *Prove the above proposition. Hint: You have to show that whenever* Q *is CE,* $A,B \subseteq M, A',B' \subseteq M'$ *are such that:* $card(A-B) = card(A'-B')$ *and* $card(A\cap B) = card(A'\cap B')$*,* $Q_M(A,B) \iff Q_{M'}(A',B')$*.*

Now assume that all universes are finite (FIN).

(0,0)

(1,0) (0,1)

(2,0) (1,1) (0,2)

(3,0) (2,1) (1,2) (0,3)

(4,0) (3,1) (2,2) (1,3) (0,4)

....

Represent CE Q by putting "+" at those (k,m) which belong to Q and "-" elsewhere.

Observation 1 *Conjunction corresponds to the intersection of* $+$*s, disjunction is the union, outer negation is the complement. Further, the inner negation of* Q *is obtained by rotating the* Q *triangle 180 degrees around the vertical axes. To get the dual of* Q*, one switches signs after the rotation.*

10 Monotone quantifiers

Definition 13 *A quantifier* Q_M *of type* $(n_1,\ldots,n_k)$ *is monotone increasing in the i'th argument iff the following holds:*

$$\text{If } \mathsf{Q}_M(R_1,\ldots,R_k) \text{ and if it holds that } R_i \subseteq R_i' \subseteq M^{n_i}, \text{ then } Q_M(R_1,\ldots,R_{i-1},R_i',R_{i+1},\ldots,R_k).$$

Definition 14 *A quantifier* Q_M *of type* $(n_1, \ldots, n_k)$ *is monotone decreasing in the i'th argument iff the following holds:*

$$\text{If } \mathsf{Q}_M(R_1, \ldots, R_k) \text{ and if it holds that } R'_i \subseteq R_i \subseteq M^{n_i}, \text{ then } Q_M(R_1, \ldots, R_{i-1}, R'_i, R_{i+1}, \ldots, R_k).$$

Particularly, for Q of type (1, 1) we have:

$\uparrow MON$ $Q_M(A, B)$ & $A \subseteq A' \subseteq M$ then $Q_M(A', B)$ (persistence).

$\downarrow MON$ $Q_M(A, B)$ & $A' \subseteq A \subseteq M$ then $Q_M(A', B)$ (anti-persistence).

$MON \uparrow$ $Q_M(A, B)$ & $B \subseteq B' \subseteq M$ then $Q_M(A, B')$.

$MON \downarrow$ $Q_M(A, B)$ & $B' \subseteq B \subseteq M$ then $Q_M(A, B')$.

Proposition 6 *Let* Q *be any type (1, 1) quantifier.* Q *is* $MON \uparrow$ *iff (a)* $\neg\mathsf{Q}$ *is* $MON \downarrow$*, (b)* $\mathsf{Q}\neg$ *is* $MON \downarrow$*, (c)* Q^d *is* $MON \uparrow$*.* Q *is* $\uparrow MON$ *iff (a)* $\neg\mathsf{Q}$ *is* $\downarrow MON$*, (b)* $\mathsf{Q}\neg$ *is* $\uparrow MON$*, (c)* Q^d *is* $\downarrow MON$*. Similarly (with reversed arrows) for the downward monotone case. In particularly, if* Q *is double monotone, its square of opposition exhibits all of the four possible doubly monotone patterns.*

Homework 6 *Prove this.*

Definition 15 *A type (1) quantifier* Q_M *is continuous (CONT) iff* $Q_M(A'), Q_M(A'')$*, where* $A' \subseteq A \subseteq A''$*, implies* $Q_M(A)$*.*

Proposition 7 (Thijsse 1983) *A quantifier is CONT iff it is the conjunction of an increasing and a decreasing quantifier.*

Hypothesis 3 *Simple NL-determiners denote CONT quantifiers.*

Proposition 8 (Westerståhl 1989) *Under FIN, persistent quantifiers satisfying CONS, EXT, and ISOM, are FO-definable.*

Homework 7 *Describe number triangles for persistent, anti-persistent and CONT CE-quantifiers of type (1,1).*

10.1 Monotonicity and inferences

(1) More than two-thirds of the students passed the exam.

(2) At least one-third of the students are athletes.

(3) Hence, at least one student who is an athlete passed the exam.

Proposition 9 *A CONS type (1, 1) quantifier* Q_M *is MON* $\uparrow$ *iff it satisfies:*

$$\mathsf{Q}_M(A, B) \wedge \mathsf{Q}^d_M(A, C) \Rightarrow \mathsf{Some}(A, B \cap C).$$

References

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