Model Transformations

Jakub Szymanik

Institute of Artificial Intelligence
University of Groningen

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Problem

Preliminaries

Ramseyification

Collectivization

Outline

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$$\mathsf{Q} \rightsquigarrow \mathsf{Q}^*$$

Outline

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Finite models

All structures are assumed to be finite.

$$\mathfrak{A} = \{\{0, \ldots, m\}, R_1, \ldots, R_r\}$$

Collections of models

Definition

Let $\tau = \{R_1, \ldots, R_r\}$ be a relational vocabulary, where R_i is l_i-ary for $1 \leq i \leq r$, and Q a class of τ-structures closed under isomorphisms. The class Q gives rise to a Lindström quantifier which we also denote by Q. The tuple $s = (l_1, \ldots, l_r)$ is the *type* of the quantifier Q.

Examples

$$\begin{aligned}
\forall &= \{(A,P) \mid P = A\}.\\
\exists &= \{(A,P) \mid P \subseteq A \mathbin{\&} P \neq \emptyset\}.\\
\text{even} &= \{(A,P) \mid P \subseteq A \mathbin{\&} \text{ card}(P) \text{ is even}\}.\\
\text{most} &= \{(A,P,S) \mid P,S \subseteq A \mathbin{\&} \text{ card}(P \cap S) > \text{card}(P - S)\}.\\
\mathrm{M} &= \{(A,P) \mid P \subseteq A \text{ and } |P| > |A|/2\}\\
\text{some} &= \{(A,P,S) \mid P,S \subseteq A \mathbin{\&} P \cap S \neq \emptyset\}.
\end{aligned}$$

Logics with Lindström quantifiers

The extension FO(Q) is defined as usual.

$$\mathfrak{A} \models Q\overline{x}_1, \ldots, \overline{x}_r\,(\phi_1(\overline{x}_1), \ldots, \phi_r(\overline{x}_r)) \text{ iff } (A, \phi_1^{\mathfrak{A}}, \ldots, \phi_r^{\mathfrak{A}}) \in Q,$$

where $\phi_i^{\mathfrak{A}} = \{\overline{a} \in A^{l_i} \mid \mathfrak{A} \models \phi_i(\overline{a})\}$

Definability

Definition

Let Q be the class of structures of type t and $\mathcal{L}$ a logic. We say that Q is *definable* in $\mathcal{L}$ if there is a sentence $\varphi \in \mathcal{L}$ of vocabulary τ_t such that for any τ_t-structure $\mathbb{M}$:

$$\mathbb{M} \models \varphi \text{ iff } \mathbb{M} \in \mathsf{Q}.$$

Elementary structures

Some structures, like $\exists^{\leq 3}$, $\exists^{=3}$, and $\exists^{\geq 3}$, are expressible in FO.

Example

$$\text{some } x\,[A(x), B(x)] \iff \exists x[A(x) \wedge B(x)].$$

Definability – Intuitions

Theorem

A Q is definable in $\mathcal{L}$ iff $\mathcal{L} \equiv \mathcal{L}(Q)$.

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Example

Question

What does it mean that, e.g. even, *is definable in* $\mathcal{L}$*?*

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Theorem

A Q *is definable in $\mathcal{L}$ iff $\mathcal{L} \equiv \mathcal{L}(Q)$.*

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Question

What does it mean that, e.g. even, *is definable in $\mathcal{L}$?*

even is definable in $\mathcal{L}$ if there is a uniform way to express even $x\ \psi(x)$ for any formula $\psi(x)$ in $\mathcal{L}$. Over a model $\mathfrak{A}$, $\psi(x)$ defines a subset $\{x \in A \mid A \models \psi(x)\}$, so the problem is to find a way to express its evenness for each $\psi(X)$.

Non-elementary structures

Theorem

'most' *and* 'even' *are not first-order definable.*

Non-elementary structures

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We can use higher-order logics:

Non-elementary structures

Theorem

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We can use higher-order logics:

Example

In $\mathbb{M} = (M, A^M, B^M)$ the sentence

$$\text{most } x\ [A(x), B(x)]$$

is true if and only if the following condition holds:

$\exists f : (A^M - B^M) \longrightarrow (A^M \cap B^M)$ such that f is injective but not surjective.

Complexity

- Finite models can be encoded as strings.
- Classes of such finite strings are languages.

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Definition

By the *complexity of* Q we mean the computational complexity of the corresponding class of finite models.

Question

$M \in \mathsf{Q}$*? (equivalently* $M \models Q$?)

Coding

Definition

Let $\tau = \{R_1, \ldots, R_k\}$ be a relational vocabulary and $\mathbb{M}$ a τ-model of the following form: $\mathbb{M} = (U, R_1^M, \ldots, R_k^M)$, where $U = \{1, \ldots, n\}$ is the universe of model $\mathbb{M}$ and $R_i^M \subseteq U^{n_i}$ is an n_i-ary relation over U, for $1 \leq i \leq k$. We define a *binary encoding for τ-models*. The code for $\mathbb{M}$ is a word over $\{0, 1, \#\}$ of length $O((\mathrm{card}(U))^c)$, where c is the maximal arity of the predicates in τ (or $c = 1$ if there are no predicates).
The code has the following form:

$$\tilde{n}\#\tilde{R_1^M}\#\ldots\#\tilde{R_n^M}, \text{ where:}$$

- $\tilde{n}$ is the part coding the universe of the model and consists of n 1s.
- $\tilde{R_i^M}$ — the code for the n_i-ary relation R_i^M — is an n^{n_i}-bit string whose j-th bit is 1 iff the j-th tuple in U^{n_i} (ordered lexicographically) is in R_i^M.
- # is a separating symbol.

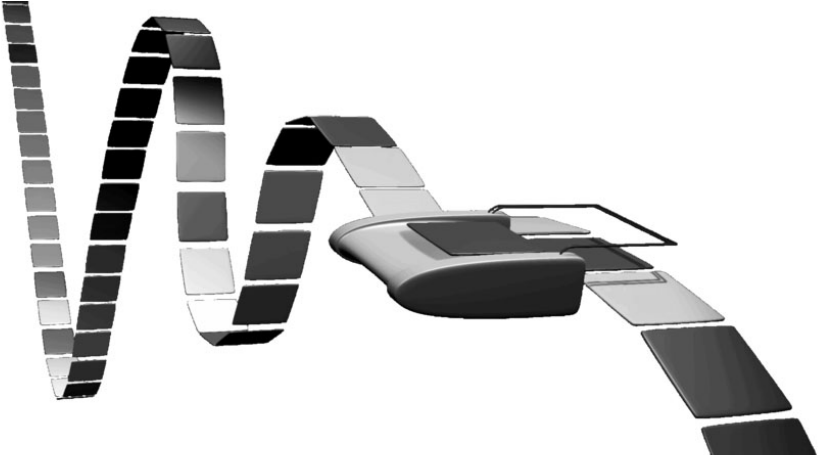
Coding Example

Consider vocabulary $\sigma = \{P, R\}$, where P is a unary predicate and R a binary relation. Take the σ-model $\mathbb{M} = (M, P^M, R^M)$, where the universe $M = \{1, 2, 3\}$, the unary relation $P^M \subseteq M$ is equal to $\{2\}$ and the binary relation $R^M \subseteq M^2$ consists of the pairs $(2, 2)$ and $(3, 2)$.

- $\tilde{n}$ consists of three 1s as there are three elements in M.
- $\tilde{P^M}$ is the string of length three with 1s in places corresponding to the elements from M belonging to P^M. Hence $\tilde{P^M} = 010$ as $P^M = \{2\}$.
- $\tilde{R^M}$ is obtained by writing down all $3^2 = 9$ binary strings of elements from M in lexicographical order and substituting 1 in places corresponding to the pairs belonging to R^M and 0 in all other places. As a result $\tilde{R^M} = 000010010$.

Adding all together the code for $\mathbb{M}$ is $111\#010\#000010010$.

What amount of resources TM needs to solve a task?



Time Complexity

Let $f : \omega \longrightarrow \omega$.

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Definition

NTIME(f), is the class of languages L for which there exists a non-deterministic Turing machine M such that for every $x \in L$ all branches in the computation tree of M on x are bounded by $f(n)$ and moreover M decides L.

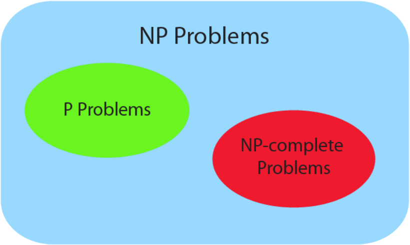
Complexity Classes P and NP

Definition

- $\text{PTIME} = \bigcup_{k \in \omega} \text{TIME}(n^k)$
- $\text{NPTIME} = \bigcup_{k \in \omega} \text{NTIME}(n^k)$

Definition

A language L is NP-complete if $L \in NP$ and every language in NP is reducible to L.


NP Problems
P Problems
NP-complete Problems

Outline

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Definition

Let Q be of type (1, 1). Define:

$$\mathrm{Ram}(\mathrm{Q})[A, R] \iff \exists X \subseteq A[\mathrm{Q}(A, X) \wedge \forall x, y \in X(x \neq y \implies R(x, y))].$$

Goal

$$\mathrm{Q} \rightsquigarrow \mathrm{Ram}(Q)$$

Cliques

$\mathrm{Ram}(\exists^{\geq k})[A, R]$ is equivalent to the following FO formula:

$$\exists x_1 \dots \exists x_k \Big[\bigwedge_{1 \leq i < j \leq k} x_i \neq x_j \;\wedge\; \bigwedge_{1 \leq i \leq k} A(x_i) \;\wedge\; \bigwedge_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k}} R(x_i, x_j) \Big].$$

Theorem

$\mathrm{Ram}(\exists^{\geq k})$ *is in LOGSPACE.*

Counting

Definition

Let $\mathbb{M} = (M, A, \ldots)$. We define:

$$\mathbb{M} \models \mathsf{C}^{\geq \mathsf{A}} x \; \varphi(x) \iff \mathrm{card}(\varphi^{\mathbb{M},x}) \geq \mathrm{card}(A).$$

Theorem

$\mathrm{Ram}(\mathsf{C}^{\geq \mathsf{A}})$ *is NP-complete.*

Proportionality

Definition

$$\mathbb{M} \models \mathrm{Q_q}[A, B] \text{ iff } \frac{\mathrm{card}(A \cap B)}{\mathrm{card}(A)} \geq q, \text{ where } 0 < q < 1 \text{ is a rational number.}$$

Theorem

If $0 < q < 1$, *then* $\mathrm{Ram}(\mathrm{Q_q})$ *is NP-complete.*

Generalization

Given $f : \omega \to \omega$, we define:

Definition

We say that a set $A \subseteq U$ is f-large relatively to U iff

$$card(A) \geq f(card(U)).$$

Definition

We define $\mathrm{R_f}$ as follows $\mathbb{M} \models \mathrm{R_f}xy\ \varphi(x, y)$ iff there is an f-large set $A \subseteq M$ such that for each $a, b \in A$, $\mathbb{M} \models \varphi(a, b)$.

Corollary

Let $f(n) = \lceil rn \rceil$, for some rational number r such that $0 < r < 1$. Then $\mathrm{R_f}$ defines NP-complete class of finite models.

Boundness

Definition

We say that a function f *is bounded* if

$$\exists m \forall n [f(n) < m \vee n - m < f(n)].$$

Otherwise, f is *unbounded*.

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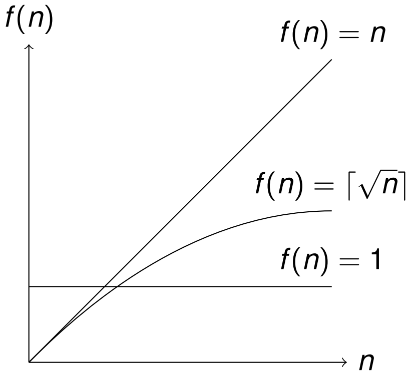

Easy Ramsey structures

Theorem

If f is PTIME computable and bounded, then the Ramsey quantifier R_f is PTIME computable.

More general observation

$$
\begin{aligned}
\exists X Q(X) \iff{} & \forall t_1 \ldots \forall t_m \forall t_{m+1} \\
& \Big[\big(\bigwedge_{1 \le i < j \le m+1} X(t_i) \implies \bigvee_{1 \le i < j \le m+1} t_i = t_j \big) \\
& \quad \vee \big(\bigwedge_{1 \le i < j \le m+1} \neg X(t_i) \implies \bigvee_{1 \le i < j \le m+1} t_i = t_j \big) \Big].
\end{aligned}
$$

This formula says that X has a property Q if and only if X consists of at most m elements or X differs from the universe on at most m elements.

Open problems

Question

Are PTIME $\mathrm{R_f}$*s exactly bounded* $\mathrm{R_f}$*s?*

Question

For what class of functions duality holds?

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Collectivization

. . . no no, not that one.

Second-order structures

Definition

Let $t = (s_1, \ldots, s_w)$, where $s_i = (l_1^i, \ldots, l_{r_i}^i)$ is a tuple of positive integers for $1 \leq i \leq w$. A second-order structure of type t is a structure of the form $(A, P_1, \ldots, P_w)$, where $P_i \subseteq \mathcal{P}(A^{l_1^i}) \times \cdots \times \mathcal{P}(A^{l_{r_i}^i})$.

Collections of second-order models

Definition

A second-order generalized quantifier $\mathcal{Q}$ of type t is a class of structures of type t such that $\mathcal{Q}$ is closed under isomorphisms.

Examples

$$
\begin{aligned}
\exists_1^2 &= \{(A, P) \mid P \subseteq \mathcal{P}(A) \;\&\; P \neq \emptyset\}. \\
\textsc{Even} &= \{(A, P) \mid P \subseteq \mathcal{P}(A) \;\&\; \mathsf{card}(P) \text{ is even}\}. \\
\textsc{Even}' &= \{(A, P) \mid P \subseteq \mathcal{P}(A) \;\&\; \forall X \in P(\mathsf{card}(X) \text{ is even})\}. \\
\textsc{Most} &= \{(A, P, S) \mid P, S \subseteq \mathcal{P}(A) \;\&\; \mathsf{card}(P \cap S) > \mathsf{card}(P - S)\}. \\
\textsc{Most}^1 &= \{(A, P) \mid P \subseteq \mathcal{P}(A) \;\&\; \mathsf{card}(P) > 2^{\mathsf{card}(A)-1}\}.
\end{aligned}
$$

FO($\mathcal{Q}$)

$$\mathfrak{A} \models \mathcal{Q}\overline{X}_1, \ldots, \overline{X}_w\,(\phi_1, \ldots, \phi_w) \text{ iff } (A, \phi_1^{\mathfrak{A}}, \ldots, \phi_w^{\mathfrak{A}}) \in \mathcal{Q},$$

where $\phi_i^{\mathfrak{A}} = \{\overline{R} \in \mathcal{P}(A^{l_1^i}) \times \cdots \times \mathcal{P}(A^{l_{r_i}^i}) \mid \mathfrak{A} \models \phi_i(\overline{R})\}$.

Warning

Do not confuse:

- FO GQs (Lindström) with FO-definable quantifiers

  E.g. most is FO GQs but is not FO-definable.
- SO GQs with SO-definable quantifiers

  E.g. MOST is SO GQs but not SO-definable.

Goal

$$\mathsf{Q} \rightsquigarrow \mathcal{Q}$$

Definability for second-order structures

Question

How do we formalize definability for SOGQs?

Definability for second-order structures

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Example

$\exists^2_1$ is definable in $\mathcal{L}$ if there is a uniform way to express $\exists^2_1 X \psi(X)$ for any formula $\psi(X)$ in $\mathcal{L}$. Over a model $\mathfrak{A}$, $\psi(X)$ defines a collection of subsets $\{C \subseteq A \mid \mathfrak{A} \models \psi(C)\}$, so the problem is to find a way to express its non-emptyness for each $\psi(X)$.

$\mathcal{L}(\mathcal{G}_1, \ldots, \mathcal{G}_w)$

Definition

Let $\mathcal{L}$ be a logic, $t = (s_1, \ldots, s_w)$ a second-order type, and let $\mathcal{G}_1, \ldots, \mathcal{G}_w$ be first-order quantifier symbols of types $s_1, \ldots, s_w$.

1. The models of $\mathcal{L}(\mathcal{G}_1, \ldots, \mathcal{G}_w)$ are of the form $\mathcal{A} = (\mathfrak{A}, G_1, \ldots, G_w)$, where $\mathfrak{A}$ is a first-order model and

$$G_i \subseteq \mathcal{P}(A^{l_1^i}) \times \cdots \times \mathcal{P}(A^{l_{r_i}^i}).$$

2. The quantifiers $\mathcal{G}_i$ are interpreted using the relations G_i:

$$\mathcal{A} \models \mathcal{G}_i \bar{x}_1, \ldots, \bar{x}_{r_i}(\phi_1(\bar{x}_1), \ldots, \phi_{r_i}(\bar{x}_{r_i}))$$

iff $(\phi_1^{\mathcal{A}}, \ldots, \phi_{r_i}^{\mathcal{A}}) \in G_i$.

Definability—definition

Observation

If $\phi \in \mathcal{L}(\mathcal{G}_1, \ldots, \mathcal{G}_w)$ is a sentence of vocabulary $\tau = \emptyset$. Then

$$\mathrm{Mod}(\phi) = \{(A, G_1, \ldots, G_w) \mid (A, G_1, \ldots, G_w) \models \phi\}$$

corresponds to a second-order generalized quantifier of type t.

Definition

Let $\mathcal{Q}$ be a quantifier of type t. The quantifier $\mathcal{Q}$ is definable in a logic $\mathcal{L}$ if there is $\phi \in \mathcal{L}(\mathcal{G}_1, \ldots, \mathcal{G}_w)$ of vocabulary $\sigma = \emptyset$ such that for any t-structure $(A, G_1, \ldots, G_w)$,

$$(A, G_1, \ldots, G_w) \models \phi \Leftrightarrow (A, G_1, \ldots, G_w) \in \mathcal{Q}.$$

Characterizing definability—main idea

Recall, $\mathcal{Q}$ of type ((1)) is definable in SO if there is a sentence $\phi \in \mathrm{SO}(\mathcal{G})$ such that for all second-order structures (A, G):

$$(A, G) \models \phi \Leftrightarrow (A, G) \in \mathcal{Q}\,.$$

We show that SO and the relation G can be replaced by FO and a unary relation P by passing from A to a domain of cardinality $2^{|A|}$.

First-order encoding of second-order structures

Observation

1. *There is a one-to-one correspondence between integers* $m \in B = \{0, \ldots, 2^n - 1\}$ *and subsets of* $A = \{0, \ldots, n-1\}$*;*
2. *Relations of A can be encoded as tuples of elements of B;*
3. *Sets of relations of A by relations of B.*

Formally

Definition

Let $t = (s_1, \ldots, s_w)$ be a type where $s_i = (1, \ldots, 1)$ is of length r_i for $1 \leq i \leq w$. Let $\mathfrak{A} = (A, G_1, \ldots, G_w)$ be a t-structure where $A = \{0, \ldots, n-1\}$ and $G_i \subseteq \mathcal{P}(A) \times \cdots \times \mathcal{P}(A)$. Denote by $\hat{\mathfrak{A}} = (B, P_1, \ldots, P_w)$ the following first-order structure of vocabulary $\tau = \{P_1, \ldots, P_w\}$, where P_i is a r_i-ary predicate, and

1. $B = \{0, \ldots, 2^n - 1\}$,
2. $P_i = \{(j_1, \ldots, j_{r_i}) \in B^{r_i} \mid (J_1, \ldots, J_{r_i}) \in G_i\}$, where, for $1 \leq k \leq r_i$, $\mathrm{bin}(j_k)$ is given by $s_0 \cdots s_{n-1}$, and $s_l = 1 \Leftrightarrow l \in J_k$.

Definition

For a quantifier $\mathcal{Q}$ of type t, we denote by $\mathcal{Q}^\star$ the first-order quantifier of vocabulary τ defined by

$$\mathcal{Q}^\star := \{\hat{\mathfrak{A}} : \mathfrak{A} \in \mathcal{Q}\},$$

where $\hat{\mathfrak{A}}$ is the first-order encoding of $\mathfrak{A}$.

Characterization

Theorem

Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be monadic quantifiers. Then $\mathcal{Q}_1$ is definable in $\mathrm{MSO}(\mathcal{Q}_2, +)$ if and only if $\mathcal{Q}_1^\star$ is definable in $\mathsf{FO}(\mathcal{Q}_2^\star, +, \times)$.

Characterization

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Built-in addition unleashes the expressive power of MSO.

Corollary: computational complexity

Theorem

If the quantifier MOST *is definable in second-order logic, then counting hierarchy,* CH *is equal polynomial hierarchy,* PH*. Moreover,* CH *collapses to its second level.*

Proof.

The logic FO(MOST) can define complete problems for each level of the CH (Kontinen&Niemisto'06). If MOST was definable in SO, then FO(MOST) $\leq$ SO and therefore SO would contain complete problems for each level of the CH. This would imply that CH $=$ PH and furthermore that CH $\subseteq$ PH $\subseteq C_2P$. $\square$

Corollary: undefinability result

Theorem

The quantifier $\textsc{Most}^1$ *is not definable in* SO*.*

Proof.

Show that definability of $\textsc{Most}^1$ in SO implies that, for some k, the quantifier M is definable in FO$(+, \times)$ over cardinalities 2^{n^k}. Over these cardinalities, we could then express PARITY in the logic FO$(+, \times)$. This contradicts the result of Ajtai(1983). □

Outlook

Question

Un(definability) theory for SOGQs.

Summary

2 case studies motivated by the formal semantics.

1. Ramsey counting structures are NP-hard.
2. Ramsey proportional structures are NP-hard.
3. Bounded Ramsey structures are in PTIME.

Question

What is the characterization of Ramsey graphs?

1. Definability of SOGQs can be reduced to that of GQs.
2. Some collective structures are not definable in SO.

Question

What is the definability theory for SOGQs?

What are other interesting transformations?

$$Q \rightsquigarrow Q^*$$

More details in:

J. Kontinen and J. Szymanik
A Remark on Collective Quantification, Journal of Logic, Language and Information, Volume 17, Number 2, 2008, pp. 131–140.

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Computational Complexity of Polyadic Lifts of Generalized Quantifiers in Natural Language, Linguistics and Philosophy, Vol. 33, Iss. 3, 2010, pp. 5–250.

J. Kontinen and J. Szymanik
Characterizing Definability of Second-Order Generalized Quantifiers, 6642, 2011, pp. 187–200.