Characterizing Definability of Second-Order Generalized Quantifiers

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Abstract

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2. Some collective quantifiers are not definable in SO.
3. Then they can not be defined via the type-shifting strategy.
4. Is it a problem for formal semantics?

Outline

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Characterizing complexity of NL-quantifiers

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2. Complexity of reasoning

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Focus on distributive readings.

Collectivity in language

(1.) All the Knights but King Arthur *met in secret*.

(2.) Most climbers *are friends*.

(3.) John and Mary *love each other*.

(4.) The samurai *were twelve in number*.

(5.) Many girls *gathered*.

(6.) Soldiers *surrounded* the Alamo.

(7.) Tikitu and Samson *lifted* the table.

Examples

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(2.) Some students played poker together.

(2'.) $\exists X[X \subseteq \text{Students} \wedge \text{Play}(X)]$.

Type-shifting strategy

1. Existential modifier (Van Der Does 1992)
2. Neutral modifier (Van Der Does 1992)
3. Determiner fitting (Winter 2001):

$$((et)((et)t)) \rightsquigarrow (((et)t)(((et)t)t))$$

Expressive power of type-shifting

Theorem

Let Q be a quantifier definable in SO*. Then the collective quantifiers Q^{EM}, Q^{N}, and Q^{dfit} are definable in* SO*.*

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Corollary

Type-shifting strategy cannot take us outside SO.

Is it enough?

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If the quantifier Most *is definable in second-order logic, then counting hierarchy,* CH *is equal polynomial hierarchy,* PH*. Moreover,* CH *collapses to its second level.*

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We characterize the definability of collective quantifiers.

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Preliminaries

- We consider finite structures. The universe of a structure $\mathfrak{A}$ is denoted by A. We assume A is of the form $\{0, \ldots, m\}$ for some $m \in \mathbb{N}$.

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- We consider logics with built-in relations. In addition to $<$, which is interpreted naturally, we use the relations $+$, $\times$, and BIT defined by: $\mathrm{BIT}(a, j)$ holds iff the bit of order 2^j is 1 in the binary representation of a.

Preliminaries

Many of the logics considered in this talk correspond to interesting complexity classes:

- $\mathrm{FO}(<, +, \times) \equiv \mathrm{LH} \equiv \mathrm{DLOGTIME} - \mathrm{uniform\ AC}^0$
- $\mathrm{MSO}(+) \equiv \mathrm{LINH}$ (over strings)
- SO $\equiv \mathrm{PH}$
- $\mathrm{FO}(\mathsf{M}, +, \times) \equiv \mathrm{LCH} \equiv \mathrm{DLOGTIME} - \mathrm{uniform\ TC}^0$
- $\mathrm{FO}(\mathsf{Most}^1, <) \equiv \mathrm{LINCH}$ (over strings)
- $\mathrm{FO}(\mathsf{Most}^k)_{k \in \mathbb{N}^*} \equiv \mathrm{CH}$
- $\mathrm{FO}(\mathsf{D}_k, +, \times) \equiv \mathrm{DLOGTIME} - \mathrm{uniform\ AC}^0[p]$

Lindström quantifiers

Definition

Let $\tau = \{P_1, \ldots, P_r\}$ be a relational vocabulary, where P_i is l_i-ary for $1 \leq i \leq r$, and Q a class of τ-structures closed under isomorphisms. The class Q gives rise to a generalized quantifier which we also denote by Q. The tuple $s = (l_1, \ldots, l_r)$ is the *type* of the quantifier Q.

Examples Lindström quantifiers

$$
\begin{array}{rcl}
\forall & = & \{(A,P) \mid P = A\}. \\
\exists & = & \{(A,P) \mid P \subseteq A \ \&\ P \neq \emptyset\}. \\
\text{even} & = & \{(A,P) \mid P \subseteq A \ \&\ |P| \text{ is even}\}. \\
\textit{most} & = & \{(A,P,S) \mid P,S \subseteq A \ \&\ |(P \cap S)| > |(P - S)|\}. \\
\mathsf{M} & = & \{(A,P) \mid P \subseteq A \text{ and } |P| > |A|/2\} \\
\text{Some} & = & \{(A,P,S) \mid P,S \subseteq A \ \&\ P \cap S \neq \emptyset\} \\
Q_S & = & \{(A,P) \mid P \subseteq A \text{ and } |P| \in S\}.
\end{array}
$$

If $S = \{kn \mid n \in \mathbb{N}\}$ for some $k \in \mathbb{N}$, we denote Q_S by D_k.

Logics with Lindström quantifiers

The extension $\mathrm{FO}(Q)$ is defined as usual.

$$\mathfrak{A} \models Q\overline{x}_1, \ldots, \overline{x}_r\,(\phi_1(\overline{x}_1), \ldots, \phi_r(\overline{x}_r)) \text{ iff } (A, \phi_1^{\mathfrak{A}}, \ldots, \phi_r^{\mathfrak{A}}) \in Q,$$

where $\phi_i^{\mathfrak{A}} = \{\overline{a} \in A^{l_i} \mid \mathfrak{A} \models \phi_i(\overline{a})\}$

Second-order structures

Definition

Let $t = (s_1, \ldots, s_w)$, where $s_i = (l_1^i, \ldots, l_{r_i}^i)$ is a tuple of positive integers for $1 \leq i \leq w$. A second-order structure of type t is a structure of the form $(A, P_1, \ldots, P_w)$, where $P_i \subseteq \mathcal{P}(A^{l_1^i}) \times \cdots \times \mathcal{P}(A^{l_{r_i}^i})$.

Second-order generalized quantifiers

Definition

A second-order generalized quantifier $\mathcal{Q}$ of type t is a class of structures of type t such that $\mathcal{Q}$ is closed under isomorphisms.

Definition

$\mathcal{Q}$ is *monadic* if $l^i_j = 1$ for all $1 \leq j \leq r_i$ and $1 \leq i \leq w$.

Examples of second-order GQs

$$
\begin{array}{rcl}
\exists^2_1 & = & \{(A,P) \mid P \subseteq \mathcal{P}(A) \;\&\; P \neq \emptyset\}. \\
\mathrm{Even} & = & \{(A,P) \mid P \subseteq \mathcal{P}(A) \;\&\; |P| \text{ is even}\}. \\
\mathrm{Even}' & = & \{(A,P) \mid P \subseteq \mathcal{P}(A) \;\&\; \forall X \in P(|X| \text{ is even})\}. \\
\mathsf{Most} & = & \{(A,P,S) \mid P,S \subseteq \mathcal{P}(A) \;\&\; |(P \cap S)| > |(P - S)|\}. \\
\mathsf{Most}^1 & = & \{(A,P) \mid P \subseteq \mathcal{P}(A) \;\&\; |P| > 2^{|A|-1}\} \\
\mathsf{Most}^k & = & \{(A,P) \mid P \subseteq \mathcal{P}(A^k) \text{ and } |P| > 2^{|A|^k-1}\} \\
\mathcal{Q}_S & = & \{(A,P) \mid P \subseteq \mathcal{P}(A) \text{ and } |P| \in S\}.
\end{array}
$$

If $S = \{kn \mid n \in \mathbb{N}\}$ for some $k \in \mathbb{N}$, we denote $\mathcal{Q}_S$ by $\mathcal{D}_k$.

FO($\mathcal{Q}$)

$$\mathfrak{A} \models \mathcal{Q}\overline{X}_1, \ldots, \overline{X}_w\,(\phi_1, \ldots, \phi_w) \text{ iff } (A, \phi_1^{\mathfrak{A}}, \ldots, \phi_w^{\mathfrak{A}}) \in \mathcal{Q},$$

where $\phi_i^{\mathfrak{A}} = \{\overline{R} \in \mathcal{P}(A^{l_1^i}) \times \cdots \times \mathcal{P}(A^{l_{r_i}^i}) \mid \mathfrak{A} \models \phi_i(\overline{R})\}$.

Warning

Do not confuse:

- FO GQs (Lindström) with FO-definable quantifiers

  E.g. most is FO GQs but is not FO-definable.
- SO GQs with SO-definable quantifiers

  E.g. Most is SO GQs but not SO-definable.

Outline

Definability—intuitions

Theorem

A first-order Q is definable in $\mathcal{L}$ iff $\mathcal{L} \equiv \mathcal{L}(Q)$.

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Question

How do we formalize definability for SOGQs?

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Example

$\exists^2_1$ is definable in $\mathcal{L}$ if there is a uniform way to express $\exists^2_1 X \psi(X)$ for any formula $\psi(X)$ in $\mathcal{L}$. Over a model $\mathfrak{A}$, $\psi(X)$ defines a collection of subsets $\{C \subseteq A \mid \mathfrak{A} \models \psi(C)\}$, so the problem is to find a way to express its non-emptyness for each $\psi(X)$.

$\mathcal{L}(\mathcal{G}_1, \ldots, \mathcal{G}_w)$

Definition

Let $\mathcal{L}$ be a logic, $t = (s_1, \ldots, s_w)$ a second-order type, and let $\mathcal{G}_1, \ldots, \mathcal{G}_w$ be first-order quantifier symbols of types $s_1, \ldots, s_w$.

1. The models of $\mathcal{L}(\mathcal{G}_1, \ldots, \mathcal{G}_w)$ are of the form $\mathcal{A} = (\mathfrak{A}, G_1, \ldots, G_w)$, where $\mathfrak{A}$ is a first-order model and

$$G_i \subseteq \mathcal{P}(A^{l^i_1}) \times \cdots \times \mathcal{P}(A^{l^i_{r_i}}).$$

2. The quantifiers $\mathcal{G}_i$ are interpreted using the relations G_i:

$$\mathcal{A} \models \mathcal{G}_i \bar{x}_1, \ldots, \bar{x}_{r_i}(\phi_1(\bar{x}_1), \ldots, \phi_{r_i}(\bar{x}_{r_i}))$$

   iff $(\phi_1^{\mathcal{A}}, \ldots, \phi_{r_i}^{\mathcal{A}}) \in G_i$.

Definability—definition

Observation

If $\phi \in \mathcal{L}(\mathcal{G}_1, \ldots, \mathcal{G}_w)$ is a sentence of vocabulary $\tau = \emptyset$. Then

$$\mathrm{Mod}(\phi) = \{(A, G_1, \ldots, G_w) \mid (A, G_1, \ldots, G_w) \models \phi\}$$

corresponds to a second-order generalized quantifier of type t.

Definition

Let $\mathcal{Q}$ be a quantifier of type t. The quantifier $\mathcal{Q}$ is definable in a logic $\mathcal{L}$ if there is $\phi \in \mathcal{L}(\mathcal{G}_1, \ldots, \mathcal{G}_w)$ of vocabulary $\sigma = \emptyset$ such that for any t-structure $(A, G_1, \ldots, G_w)$,

$$(A, G_1, \ldots, G_w) \models \phi \Leftrightarrow (A, G_1, \ldots, G_w) \in \mathcal{Q}.$$

Definability— some basic facts

Theorem (Kontinen 2010)

If $\mathcal{Q}$ is definable in $\mathcal{L}$ then $\mathcal{L} \equiv \mathcal{L}(\mathcal{Q})$.

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If $\mathcal{Q}$ is definable in $\mathcal{L}$ then $\mathcal{L} \equiv \mathcal{L}(\mathcal{Q})$.

Theorem (Kontinen 2010)

There is a quantifier $\mathcal{Q}$ of type $((1))$ which is not definable in FO and satisfies $\mathrm{FO} \equiv \mathrm{FO}(\mathcal{Q})$.

Characterizing definability—main idea

Recall, $\mathcal{Q}$ of type ((1)) is definable in SO if there is a sentence $\phi \in \mathsf{SO}(\mathcal{G})$ such that for all second-order structures (A, G):

$$(A, G) \models \phi \Leftrightarrow (A, G) \in \mathcal{Q}\,.$$

We show that SO and the relation G can be replaced by FO and a unary relation P by passing from A to a domain of cardinality $2^{|A|}$.

First-order encoding of second-order structures

Observation

1. *There is a one-to-one correspondence between integers* $m \in B = \{0, \ldots, 2^n - 1\}$ *and subsets of* $A = \{0, \ldots, n - 1\}$*;*
2. *Relations of* A *can be encoded as tuples of elements of* B*;*
3. *Sets of relations of* A *by relations of* B*.*

Formally

Definition

Let $t = (s_1, \ldots, s_w)$ be a type where $s_i = (1, \ldots, 1)$ is of length r_i for $1 \leq i \leq w$. Let $\mathfrak{A} = (A, G_1, \ldots, G_w)$ be a t-structure where $A = \{0, \ldots, n-1\}$ and $G_i \subseteq \mathcal{P}(A) \times \cdots \times \mathcal{P}(A)$. Denote by $\hat{\mathfrak{A}} = (B, P_1, \ldots, P_w)$ the following first-order structure of vocabulary $\tau = \{P_1, \ldots, P_w\}$, where P_i is a r_i-ary predicate, and

1. $B = \{0, \ldots, 2^n - 1\}$,
2. $P_i = \{(j_1, \ldots, j_{r_i}) \in B^{r_i} \mid (J_1, \ldots, J_{r_i}) \in G_i\}$, where, for $1 \leq k \leq r_i$, $\mathrm{bin}(j_k)$ is given by $s_0 \cdots s_{n-1}$, and $s_l = 1 \Leftrightarrow l \in J_k$.

Definition

For a quantifier $\mathcal{Q}$ of type t, we denote by $\mathcal{Q}^\star$ the first-order quantifier of vocabulary τ defined by

$$\mathcal{Q}^\star := \{\hat{\mathfrak{A}} : \mathfrak{A} \in \mathcal{Q}\},$$

where $\hat{\mathfrak{A}}$ is the first-order encoding of $\mathfrak{A}$.

Characterization

Theorem

Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be monadic quantifiers. Then $\mathcal{Q}_1$ is definable in $\mathrm{MSO}(\mathcal{Q}_2, +)$ if and only if $\mathcal{Q}_1^\star$ is definable in $\mathrm{FO}(\mathcal{Q}_2^\star, +, \times)$.

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Built-in addition unleashes the expressive power of MSO.

Corollaries

Definition

Let $t = (s_1, \ldots, s_w)$ and τ be as before. Let $\mathcal{Q}$ be of type t. The quantifier $\mathcal{Q}$ is *numerical* if there is $T \subseteq \mathbb{N}^w$ s.t. for all $(A, P_1, \ldots, P_w)$

$$(A, P_1, \ldots, P_w) \in \mathcal{Q} \Leftrightarrow (|P_1|, \ldots, |P_w|) \in T.$$

We denote $\mathcal{Q}$ by $\mathcal{Q}_T$ and by Q_T the first-order numerical quantifier (defined analogously) of vocabulary τ.

For a numerical $\mathcal{Q}_T$, the quantifier $\mathcal{Q}^\star_T$ is just the restriction of Q_T to the cardinalities 2^n:

$$\mathcal{Q}^\star_T = \{(A, P_1, \ldots, P_w) \in Q_T : |A| = 2^n \text{ for some } n \in \mathbb{N}\}.$$

Corollaries cont.

Theorem

Let $\mathcal{Q}_T$ be a numerical quantifier and $k \in \mathbb{N}$. Then

1. *$\mathcal{Q}_T$ is definable in $\mathrm{MSO}(+)$ iff Q_T is definable in $\mathrm{FO}(+, \times)$.*
2. *$\mathcal{Q}_T$ is definable in $\mathrm{MSO}(\mathcal{D}_k, +)$ iff Q_T is definable in $\mathrm{FO}(\mathsf{D}_k, +, \times)$.*
3. *$\mathcal{Q}_T$ is definable in $\mathrm{MSO}(\mathsf{Most}^1, +)$ iff Q_T is definable in $\mathrm{FO}(\mathsf{M}, +, \times)$.*

Most^1 is not definable in SO

Theorem

The quantifier Most^1 *is not definable in* SO*.*

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Proof.

Show that definability of Most^1 in SO implies that, for some k, the quantifier M is definable in $\mathrm{FO}(+, \times)$ over cardinalities 2^{n^k}. Over these cardinalities, we could then express PARITY in the logic $\mathrm{FO}(+, \times)$. This contradicts the result of Ajtai(1983). $\square$

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Corollary
The type-shifting strategy is not general enough to cover all collective quantification in natural language.

Outline

- Does Most[1] belong to everyday language?
  - Everyday language doesn't realize prop. col. qua.
  - No need to extend the higher-order approach to prop. qua.

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  - Everyday language doesn't realize prop. col. qua.
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Question

Did we just encounter an example where complexity restricts the expressibility of everyday language?

Summary

- Definability of SOGQs can be reduced to that of GQs.
- Most^1 is not definable in SO.
- Type-shifting strategy is restricted.
- Does NL go beyond SO?

More details in:

J. Kontinen and J. Szymanik
A Remark on Collective Quantification,
Journal of Logic, Language and Information , Volume 17, Number 2, 2008, pp. 131–140.

J. Kontinen and J. Szymanik
Characterizing Definability of Second-Order Generalized Quantifiers,
LNAI, 6642, 2011, pp. 187–200.

J. Szymanik
Quantifiers in TIME and SPACE,
PhD Dissertation, University of Amsterdam.