Complexity of backward induction games

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Outline

Only surprising thing about the WikiLeaks revelations is that they contain no surprises. Didn't we learn exactly what we expected to learn? The real disturbance was at the level of appearances: we can no longer pretend we don't know what everyone knows we know. This is the paradox of public space: even if everyone knows an unpleasant fact, saying it in public changes everything.

(Slavoj Žižek "Good Manners in the Age of WikiLeaks")

Outline

Logic and CogSci?

Question

What can logic do for CogSci, and vice versa?

Marr's levels of explanation

1. computational level:
   - problems that a cognitive ability has to overcome

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1. computational level:
   - problems that a cognitive ability has to overcome
2. algorithmic level:
   - the algorithms that may be used to achieve a solution
3. implementation level:
   - how this is actually done in neural activity

Marr, Vision: a computational investigation into the human representation and processing of the visual information, 1983

Between computational and algorithmic level

Claim

Logic can inform us about inherent properties of the problem.

Level 1,5 Complexity level:

- complexity of the possible algorithms

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Gierasimczuk et al., Logical and psychological analysis of deductive mastermind, 2012

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Example

The easier the algorithm the easier quantifier verification.

Szymanik & Zajenkowski, Comprehension of simple quantifiers, 2010

Logic and social cognition

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1. Higher-order reasonings: ‘I believe that Ann knows that Ben thinks . . . ’

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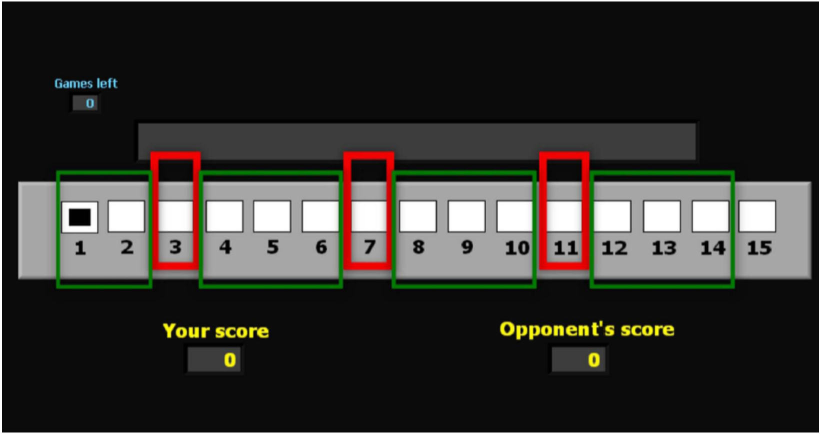
1. Higher-order reasonings: 'I believe that Ann knows that Ben thinks . . . '
2. Interacts with game-theory
3. Backward induction: tells us which sequence of actions will be chosen by agents that want to maximize their own payoffs, assuming common knowledge of rationality.
4. BI games have been extensively studied in psychology

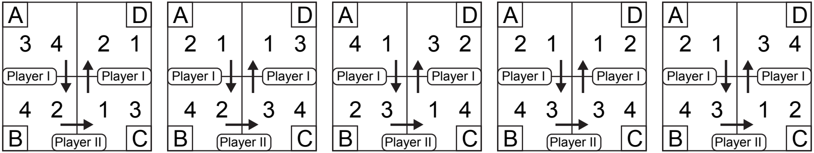
HIT-N Game



Gneezy et al. Experience and insight in the race game, 2010

Hawes et al. Experience and abstract reasoning in learning backward induction, 2012

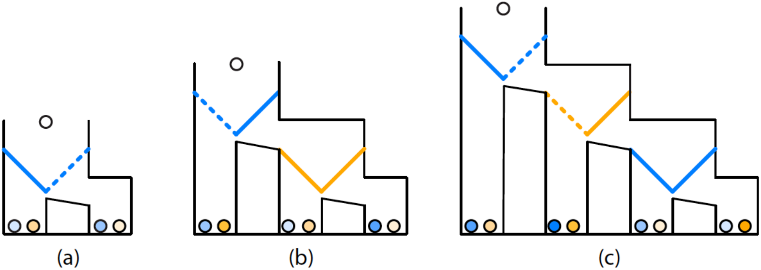
Matrix game



(a) (b) (c) (d) (e)

Hedden & Zhang What do you think I think you think?, 2002

Marble Drop Game



Meijering et al., The facilitative effect of context on second-order social reasoning, 2010

BI algorithm

At the end of the game, players have their values marked. At the intermediate stages, once all follow-up stages are marked, the player to move gets her maximal value that she can reach, while the other, non-active player gets his value in that stage.

Project

1. What is the complexity of the computational problem?
2. What makes certain trials harder than others?

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$\hookrightarrow$ human reasoning strategies and bounded rationality

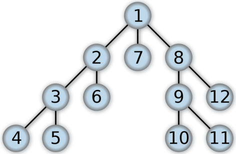
Outline

Finite finitely branching trees

$$
\begin{tikzcd}
 & \text{s,1} \arrow[dl, "\text{l}"'] \arrow[dr, "\text{r}"] & & & \\
\text{(t1, t2)} & & \text{t,2} \arrow[dl, "\text{l}"'] \arrow[dr, "\text{r}"] & & \\
 & \text{(s1, s2)} & & \text{u,1} \arrow[dl, "\text{l}"'] \arrow[dr, "\text{r}"] & \\
 & & \text{(p1, p2)} & & \text{(q1, q2)}
\end{tikzcd}
$$

BI is computable in polynomial time

- Recursive depth first-traversal of the game tree.



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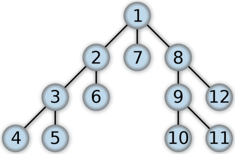


- Therefore, $BI \in PTIME$.

Question

Is BI PTIME-complete?

Question

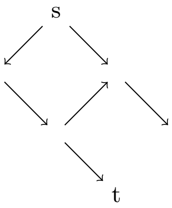
Descriptive complexity analysis of BI?

Van Benthem & Gheerbrant, Game solution, epistemic dynamics and fixed-point logics, 2010

Preliminaries: reachability

Question

Is t reachable from s?



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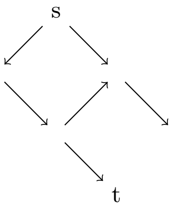

Theorem

Reachability is NL-complete.

Alternating graphs

Definition

Let an alternating graph $G = (V, E, A)$ be a directed graph whose vertices, V, are labeled universal or existential. $A \subseteq V$ is the set of universal vertices. $E \subseteq V \times V$ is the edge relation.

$$
\begin{tikzcd}
 & \mathrm{A} \arrow[dl] \arrow[dr] & & \\
\mathrm{E} \arrow[dr] & & \mathrm{E} \arrow[dr] & \\
 & \mathrm{A} \arrow[ur] \arrow[dr] & & \mathrm{A} \\
 & & \mathrm{A} &
\end{tikzcd}
$$

Reachability on alternation graphs

Definition

Let $G = (V, E, A, s, t)$ be an alternating graph. We say that t is reachable from s iff $P_a^G(s, t)$, where $P_a^G(x, y)$ is the smallest relation on vertices of G satisfying:

1. $P_a^G(x, x)$
2. If x is existential and $P_a^G(z, y)$ holds for some edge (x, z) then $P_a^G(x, y)$.
3. If x is universal, there is at least one edge leaving x, and $P_a^G(z, y)$ holds for all edges (x, z) then $P_a^G(x, y)$.

Is there an alternating path from s to t?

$$
\begin{tikzcd}
 & \text{s, A} \arrow[dl] \arrow[dr] & & \\
\text{E} \arrow[dr] & & \text{E} \arrow[dr] & \\
 & \text{A} \arrow[ur] \arrow[dr] & & \text{A} \\
 & & \text{t, A} &
\end{tikzcd}
$$

Reachability on alternating graphs is PTIME-complete

Definition
$REACH_a = \{G | P_a^G(s,t)\}$

Theorem
$REACH_a$ is PTIME-complete via first-order reductions.

Corollary on competitive games

Observation

Given G and s, $REACH_a$ intuitively corresponds to the question:
'Is s a winning position for the first player in the zero-sum game G?'

Corollary

BI for zero-sum games is PTIME-complete.

Extensive form game graphs

Definition

A two player game $G = (V, E, V_1, V_2, f_1, f_2, s, t)$ is a graph, where V is the set of nodes, $E \subseteq V \times V$ is the edge relation (available moves). For $i = 1, 2$, $V_i \subseteq V$ is the set of nodes controlled by Player i, and $V_1 \cap V_2 = \emptyset$. Finally, $f_i : V \longrightarrow \mathbb{N}$ assigns pay-offs for Player i.

BI accessibility relation

Definition

Let G be a two player game. We define the backward induction accessibility relation on G. Let $P^G_{bi}(x, y)$ be the smallest relation on vertices of G such that:

1. $P^G_{bi}(x, x)$
2. Take $i = 1, 2$. Assume that $x \in V_i$ and $P^G_{bi}(z, y)$. If the following two conditions hold, then also $P^G_{bi}(x, y)$ holds:
   - 2.1 $E(x, z)$;
   - 2.2 there is no w, v such that $E(x, w)$, $P^G_{bi}(w, v)$, and $f_i(v) > f_i(y)$.

And now, is t BI-accessible from s?

$$
\begin{tikzcd}
& s,\ 2 \arrow[dl] \arrow[dr] & & \\
1 \arrow[dr] & & 1 \arrow[dr] & \\
& 2 \arrow[ur] \arrow[dr] & & (4,\ 7) \\
& & t,\ (5,\ 6) &
\end{tikzcd}
$$

BI decision problem

Definition
$REACH_{bi} = \{G | P^G_{bi}(s,t)\}$

Theorem
$REACH_{bi}$ is PTIME-complete via first-order reductions.

Is it interesting?

- Cobham-Edmonds thesis: PTIME = tractable

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- Difficult to effectively parallelize (outside NC).
- Difficult to solve in limited space (outside L).

Outline

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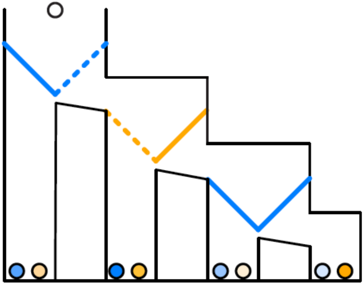
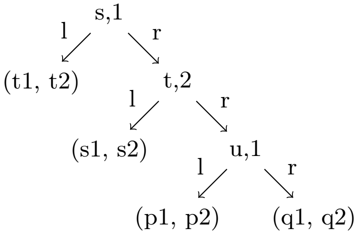


MDG decision trees



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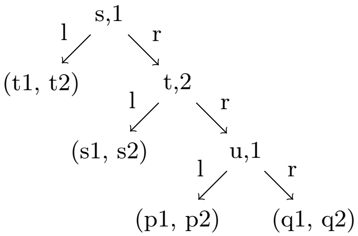

Definition

G is generic, if for each player, distinct end nodes have different pay-offs.

Question

Question

How to approximate the complexity of a single instance?

Alternation type

Definition

Let's assume that the players strictly alternate in the game. Then:

1. In a Λ_1^i tree all the nodes are controlled by Player i.
2. In a Λ_k^i tree, k-alternations, starts with an ith Player node.

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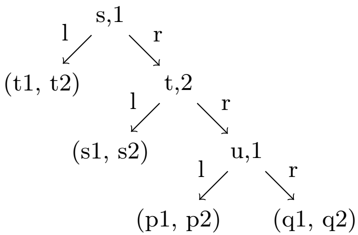

Figure: Λ_3^1 -tree

Alternation hierarchy

Definition

Let $\Lambda_k^i - REACH_{bi}$ be the $REACH_{bi}$ problem over Λ_k^i-graphs and:

$$\Lambda - REACH_{bi} = \bigcup_{i=1,2;0\leq k\leq n;n\in\omega} \Lambda_k^i - REACH_{bi}$$

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Question

Does for every $i, j \in \{1, 2\}$, the computational complexity of $REACH_{bi}$ for all Λ_{k+1}^i graphs is greater than for all Λ_k^j graphs, and all Λ_k^i graphs are of the same complexity?

Logarithmic hierarchy, LH

Definition

LH = ATIME-ALT[log n, O(1)] – the set of boolean queries computed by alternating Turing machines in O[log n] time, making a bounded number of alternations.

Theorem

$LH = FO$

Open problem

Fact

$\Lambda^i_1 - REACH_{bi} = Reachability$

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Does it correspond to logarithmic hierarchy?

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Conjecture

$\Lambda - REACH_{bi} = LH = FO$

Conjecture

$\Lambda_k^i - REACH_{bi} = ATIME - ALT[log\ n, k]$

Let's talk psychology . . .

Subjects strategies

To explain eye-tracking data: forward induction with backward reasoning.

Ghosh & Meijering On combining cognitive and formal modelling: a case study involving strategic reasoning, 2011

Λ^1_3 trees

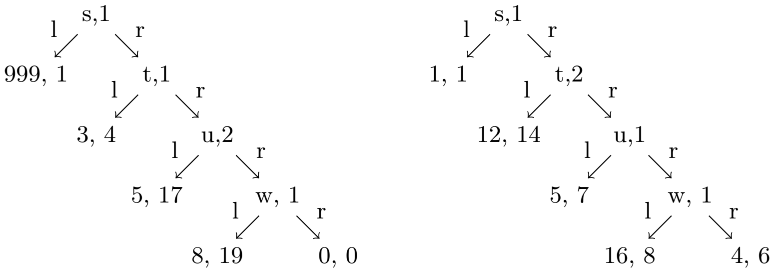

Figure: Two Λ^1_3 trees.

T^-

Definition

If T is a generic game tree with the root node controlled by Player 1 (2) and n is the highest pay-off for Player 1 (2), then T^- is the minimal subtree of T containing the root node and the node with pay-off n for Player 1 (2).

T^--example

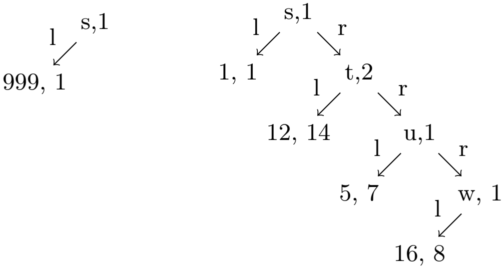

Figure: Λ_1^1 tree and Λ_3^1 tree

Experimental Conjecture

Let us take two MDG trials T_1 *and* T_2. T_1 *is easier than* T_2 *if and only if* T_1^- *is lower in the tree alternation hierarchy than* T_2^-.

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Question

What if the player doesn't control the node leading to the highest pay-off?

Other possibility: opponent types

Assume that your opponent is:

1. Predictive
2. Risk-averse
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Example of T^{risky}

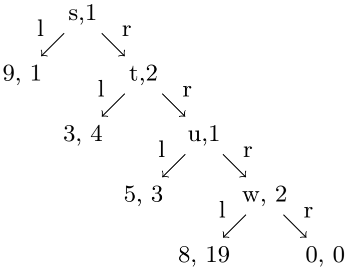

Figure: T and corresponding T^{risky}.

Example of $T^{cautious}$

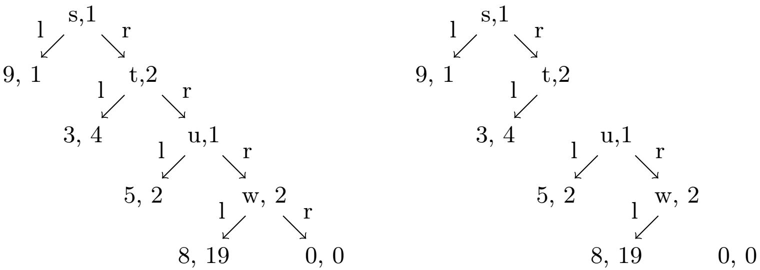

Figure: T and corresponding $T^{cautious}$.

Order-reducing strategy

Observation

Every T^{risk} *and* $T^{cautious}$ *tree is* Λ^i_1.

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Note

Resembles meaning shifts to avoid intractable interpretations ($\varphi \implies \psi$)

Mostowski & Szymanik, Semantic bounds for everyday language, 2012

Szymanik, Computational complexity of polyadic lifts of generalized quantifiers in NL, 2010

Gierasimczuk & Szymanik, Branching quantification vs. two-way quantification, 2009

New rationality concepts for bounded agents

Theorem

BI-solution is a subgame perfect equilibrium, i.e., it represents a Nash equilibrium of every subgame of the original game.

$\hookrightarrow$ agents with restricted horizon should still play BI

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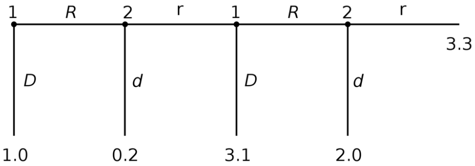
If BI is even rational in the first place . . .



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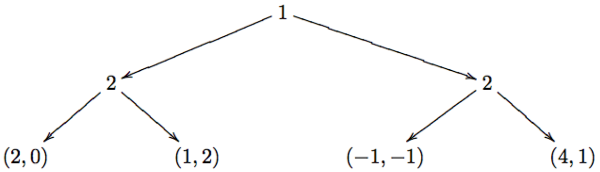
- Show correspondence with Λ_k^i-hierarchy.
- Build proof-system.
- Define proof-depth that corresponds to the reasoning difficulty.

General picture

$$\Lambda \sim LH \sim depth(\varphi) \sim |proof|$$

Example



A proof:

1. $turn_2 \wedge \langle 2\rangle(u_2 = 0 \wedge u_1 = 2) \wedge \langle 2\rangle(u_2 = 2 \wedge u_1 = 1) \wedge (2 > 1)$ (premise)
2. $turn_2 \wedge \langle 2\rangle(u_2 = -1 \wedge u_1 = -1) \wedge \langle 2\rangle(u_2 = 1 \wedge u_1 = 4) \wedge (2 > 1)$ (premise)
3. $(u_2 = 2 \wedge u_1 = 1)$ (from 1)
4. $(u_2 = 1 \wedge u_1 = 4)$ (from 2)
5. $(u_1 = 1 \wedge u_2 = 2)$ (from 3)
6. $(u_1 = 4 \wedge u_2 = 1)$ (from 4)
7. $turn_1 \wedge \langle 1\rangle(u_1 = 1 \wedge u_2 = 2) \wedge \langle 2\rangle((u_1 = 4 \wedge u_2 = 1) \wedge (4 > 1)$ (from 5, 6)
8. $(u_1 = 4 \wedge u_2 = 1)$ (from 2) (from 7)

Broader question

Question

What is the rationality theory of computationally bounded agents?