Almost All Complex Quantifiers are Simple

Jakub Szymanik

MoL 2009

Outline

Outline

Generalized Quantifier Theory

- Quantifiers occur whenever we speak.
- They influence language expressivity.
- Classical GQT studies definability issues.

Computational Complexity of Quantifiers

- How much resources is needed for processing?

Computational Complexity of Quantifiers

- How much resources is needed for processing?
- Model-checking is a part of comprehension.

Computational Complexity of Quantifiers

- How much resources is needed for processing?
- Model-checking is a part of comprehension.
- Input: $\mathbb{M}$, φ. Output: $\varphi^{\mathbb{M}}$.

Computational Complexity of Quantifiers

- How much resources is needed for processing?
- Model-checking is a part of comprehension.
- Input: $\mathbb{M}$, φ. Output: $\varphi^{\mathbb{M}}$.
- W.r.t. to model size.

Computational Complexity of Quantifiers

- How much resources is needed for processing?
- Model-checking is a part of comprehension.
- Input: $\mathbb{M}$, φ. Output: $\varphi^{\mathbb{M}}$.
- W.r.t. to model size.
- Restriction to finite models.

Background motivations

- Computational approach to cognition.
  - Cognitive task is a computational task.

Background motivations

- Computational approach to cognition.
  - Cognitive task is a computational task.
- Algorithmic theory of meaning.
  - Meaning = procedure computing denotation.

Monadic GQs

1. They are easy to compute: FA, PDA.
2. Computational model is neuropsychologically plausible.

Monadic GQs

1. They are easy to compute: FA, PDA.
2. Computational model is neuropsychologically plausible.

Question

What about computational complexity of polyadic quantifiers?

Outline

GQs

Definition

Let $t = (n_1, \ldots, n_k)$ be a k-tuple of positive integers.
A generalized quantifier of type t is a class Q of models of a vocabulary $\tau_t = \{R_1, \ldots, R_k\}$, such that R_i is n_i-ary for $1 \leq i \leq k$, and Q is closed under isomorphisms.

GQs

Definition

Let $t = (n_1, \ldots, n_k)$ be a k-tuple of positive integers.
A generalized quantifier of type t is a class Q of models of a vocabulary $\tau_t = \{R_1, \ldots, R_k\}$, such that R_i is n_i-ary for $1 \leq i \leq k$, and Q is closed under isomorphisms.

Definition

If in the above definition for all i: $n_i \leq 1$, then we say that a quantifier is *monadic*, otherwise we call it *polyadic*.

GQs as classes of models

$$
\begin{aligned}
\forall &= \{(M, P) \mid P = M\}. \\
\exists &= \{(M, P) \mid P \subseteq M \;\&\; P \neq \emptyset\}. \\
\text{even} &= \{(M, P) \mid P \subseteq M \;\&\; \text{card}(P) \text{ is even}\}. \\
\text{most} &= \{(M, P, S) \mid P, S \subseteq M \;\&\; \text{card}(P \cap S) > \text{card}(P - S)\}. \\
\text{some} &= \{(M, P, S) \mid P, S \subseteq M \;\&\; P \cap S \neq \emptyset\}.
\end{aligned}
$$

Quantifiers in finite models

- Finite models can be encoded as strings.
- GQs as classes of such finite strings are languages.

Quantifiers in finite models

- Finite models can be encoded as strings.
- GQs as classes of such finite strings are languages.

Definition

By the *complexity of a quantifier* Q we mean the computational complexity of the corresponding class of finite models.

Question

$M \in$ Q? equivalently $M \models Q$?

Outline

Possibly Branching Sentences

1. Most villagers and most townsmen hate each other.
2. One third of villagers and half of townsmen hate each other.
3. 5 villagers and 7 townsmen hate each other.

Branching Reading

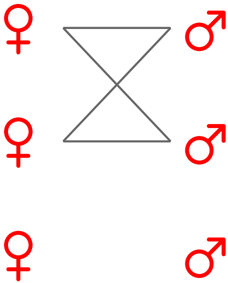
- Most girls and most boys hate each other.

$$\begin{array}{l} \text{most } x : G(x) \\ \text{most } y : B(y) \end{array} H(x, y).$$

$$\exists A \exists A' [\text{most}(G, A) \wedge \text{most}(B, A') \wedge \forall x \in A\, \forall y \in A'\ H(x, y)].$$

Illustration

- Most girls and most boys hate each other.



Branching Readings are Intractable

Theorem

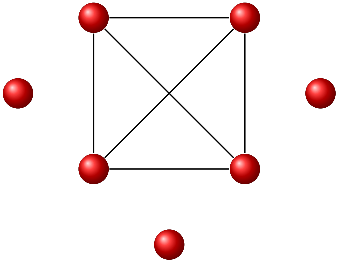
Proportional branching sentences are NP-complete.

Potentially Strong Reciprocal Sentences

1. Andi, Jarmo and Jakub laughed at one another.
2. 15 men are hitting one another.
3. Most of the PMs refer to each other.

Strong Reading

- Most of the PMs refer to each other.



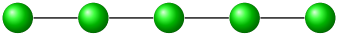
Strong Reciprocity is Intractable

Theorem

Model-checking for strong reciprocal sentences with proportional quantifiers is NP-complete.

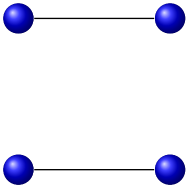
Intermediate Reading

- Most Boston pitchers sat alongside each other.



Weak Reading

- Some pirates were staring at each other in surprise.



Complexity Dichotomy

As opposed to the strong case:

Complexity Dichotomy

As opposed to the strong case:

Theorem

If Q *is PTIME, then also* $\mathrm{Ram}_{\mathrm{I}}(\mathrm{Q})$ *and* $\mathrm{Ram}_{\mathrm{W}}(\mathrm{Q})$ *are in PTIME.*

Boolean Combinations

1. At least 5 or at most 10 departments can win EU grants.
2. Between 100 and 200 students run in the marathon.
3. Not all students passed.
4. All students did not pass.

Boolean Combinations

Definition

Let Q, Q′ be generalized quantifiers, both of type $(n_1, \dots, n_k)$.
We define:

$$(\mathsf{Q} \wedge \mathsf{Q}')_M[R_1, \dots, R_k] \iff \mathsf{Q}_M[R_1, \dots, R_k] \text{ and } \mathsf{Q}'_M[R_1, \dots, R_k]$$

$$(\mathsf{Q} \vee \mathsf{Q}')_M[R_1, \dots, R_k] \iff \mathsf{Q}_M[R_1, \dots, R_k] \text{ or } \mathsf{Q}'_M[R_1, \dots, R_k]$$

$$(\neg\mathsf{Q})_M[R_1, \dots, R_k] \iff \text{not } \mathsf{Q}_M[R_1, \dots, R_k]$$

$$(\mathsf{Q}\neg)_M[R_1, \dots, R_k] \iff Q_M[R_1, \dots, R_{k-1}, M - R_k]$$

Boolean Operations are Tractable

Theorem

Let Q *and* Q' *be generalized quantifiers computable in polynomial time with respect to the size of a universe. Then the quantifiers:* (1) $\neg Q$*;* (2) $Q\neg$*;* (3) $Q \wedge Q'$ *are PTIME computable.*

Iteration

1. Most logicians criticized some papers.
2. $\mathrm{It}(\mathrm{most}, \mathrm{some})$[Logicians, Papers, Criticized].

Definition

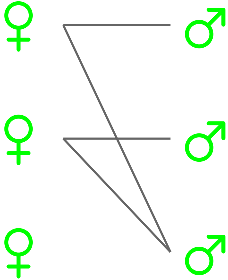
Let Q and Q′ be generalized quantifiers of type (1, 1). Let A, B be subsets of the universe and R a binary relation over the universe. Suppressing the universe, we will define the *iteration* operator as follows:

$$\mathrm{It}(\mathrm{Q}, \mathrm{Q}')[A, B, R] \iff \mathrm{Q}[A, \{a \mid \mathrm{Q}'[B, R_{(a)}]\}],$$

where $R_{(a)} = \{b \mid R(a, b)\}$.

Illustration

- Most girls and most boys hate each other.



Iteration is easy

Theorem

Let Q *and* Q' *be generalized quantifiers computable in PTIME with respect to the size of a universe. Then the quantifier* $\mathrm{It}(\mathrm{Q}, \mathrm{Q}')$ *is also PTIME computable.*

Cumulation

- Eighty professors taught sixty courses at ESSLLI'08.

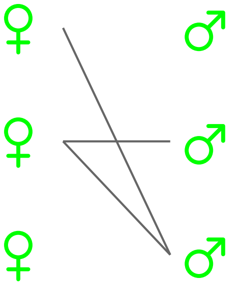
Definition

$\mathrm{Cum}(\mathrm{Q}, \mathrm{Q}')[A, B, R] \iff$

$$\mathrm{It}(\mathrm{Q}, \mathrm{some})[A, B, R] \wedge \mathrm{It}(\mathrm{Q}', \mathrm{some})[B, A, R^{-1}]$$

Illustration

- Most girls and most boys hate each other.



Cumulation is easy

Theorem

Let Q *and* Q' *be generalized quantifiers computable in PTIME with respect to the size of a universe. Then the quantifier* $\mathrm{Cum}(\mathrm{Q}, \mathrm{Q}')$ *is PTIME computable.*

Resumption

- Most twins never seperate.

Definition

Let Q be any monadic quantifier with n arguments, U a universe, and $R_1, \ldots, R_n \subseteq U^k$ for $k \geq 1$. We define the *resumption* operator as follows:

$$\mathrm{Res}^k(\mathrm{Q})_U[R_1, \ldots, R_n] \iff (\mathrm{Q})_{U^k}[R_1, \ldots, R_n].$$

Resumption is easy

Theorem

Let Q *and* Q' *be generalized quantifiers computable in PTIME with respect to the size of a universe. Then the quantifier* $\mathsf{Res}(\mathsf{Q}, \mathsf{Q}')$ *is PTIME computable.*

Outline

Basic Operations are Tractable

Theorem

Let Q *and* Q' *be generalized quantifiers computable in polynomial time with respect to the size of a universe. Then the quantifiers:* (1) $\neg\mathsf{Q}$*;* (2) $\mathsf{Q}\neg$*;* (3) $\mathsf{Q} \wedge \mathsf{Q}'$*;* (4) $\mathrm{It}(\mathsf{Q}, \mathsf{Q}')$*;* (5) $\mathrm{Cum}(\mathsf{Q}, \mathsf{Q}')$*;* (6) $\mathrm{Res}(\mathsf{Q})$ *are PTIME computable.*

Take home message

Everyday simple determiners in NL are in PTIME.

Take home message

Everyday simple determiners in NL are in PTIME.
PTIME quantifiers are closed under the common polyadic lifts.

Take home message

Everyday simple determiners in NL are in PTIME.
PTIME quantifiers are closed under the common polyadic lifts.

Common polyadic quantifiers in NL are tractable.

Thank you for attention. Questions?