A Dichotomy Result for Ramsey Quantifiers

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Joint work with Ronald de Haan

Outline

Quantifier expressivity/complexity trade-off

Everyone knows each other here.

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Marek
Joanna
Konrad
Leszek
Marcin
Darek
Michał
Nina
Jakub
Amelie
Cezary

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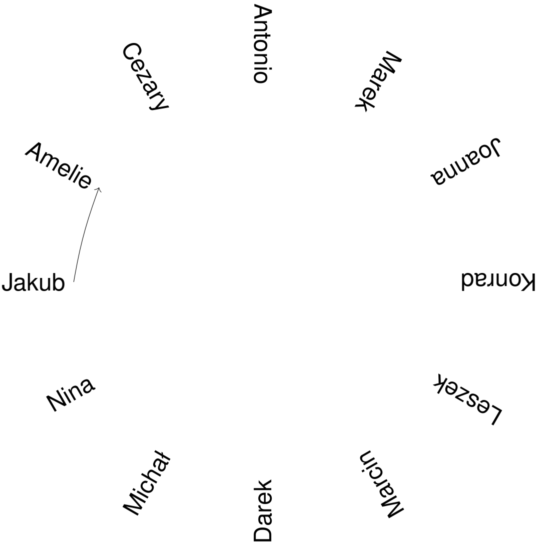

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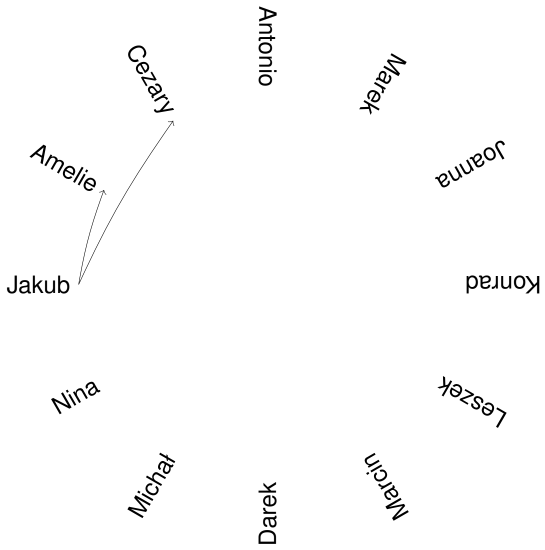

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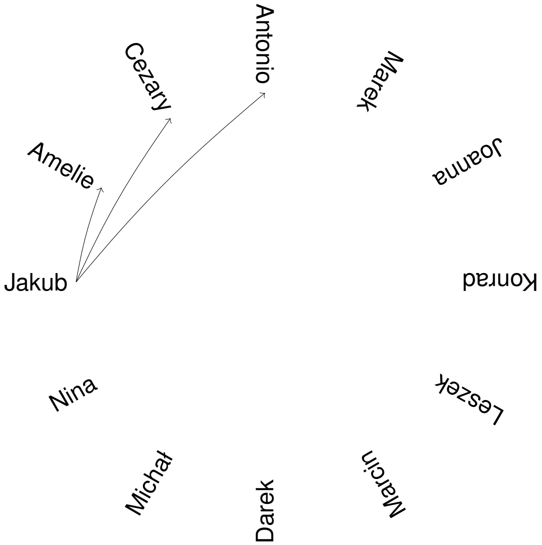

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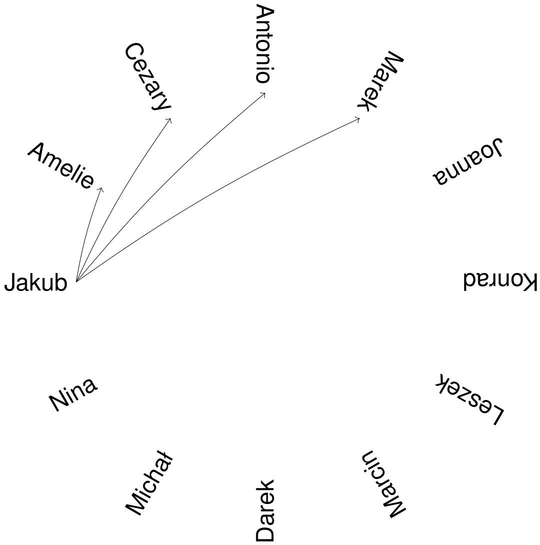

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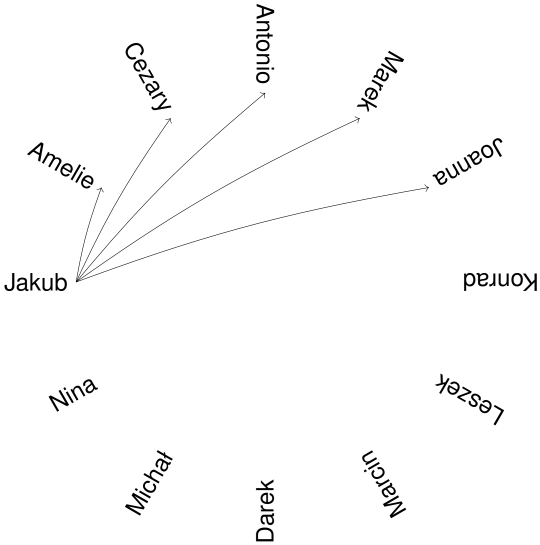

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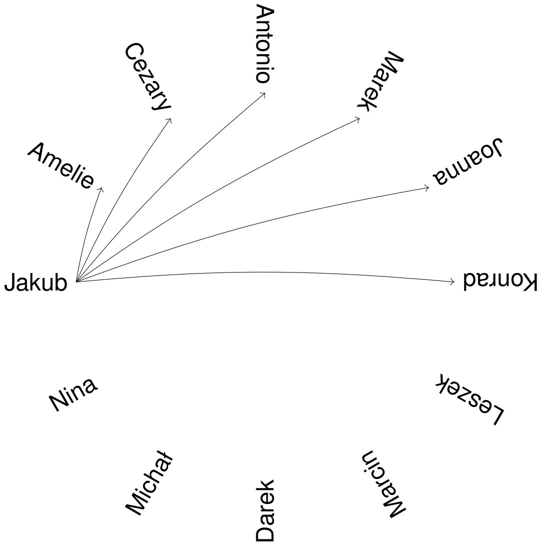

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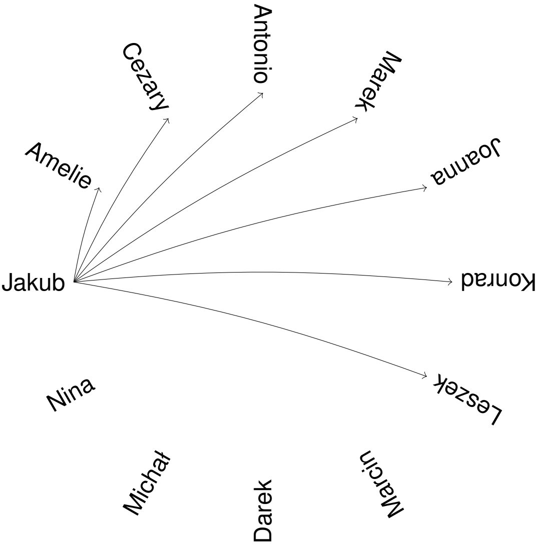

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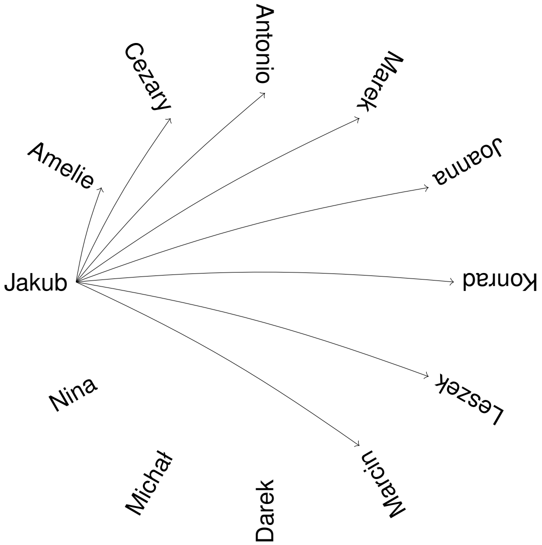

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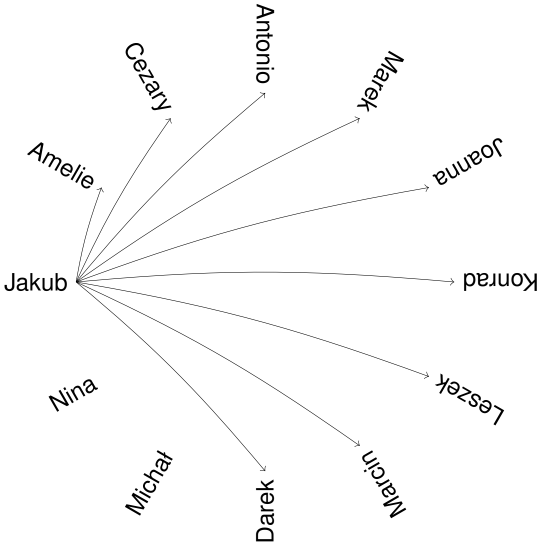

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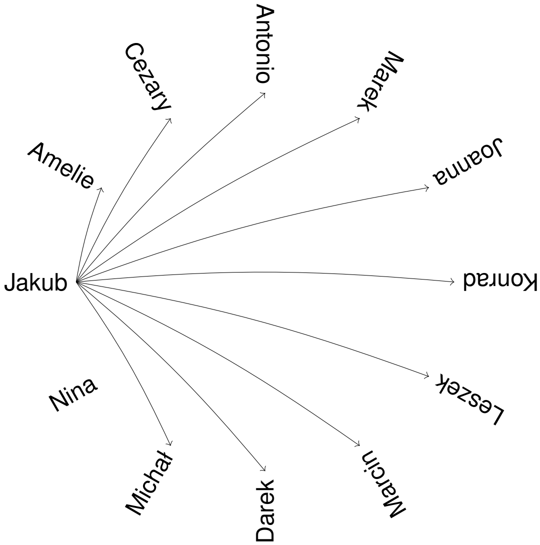

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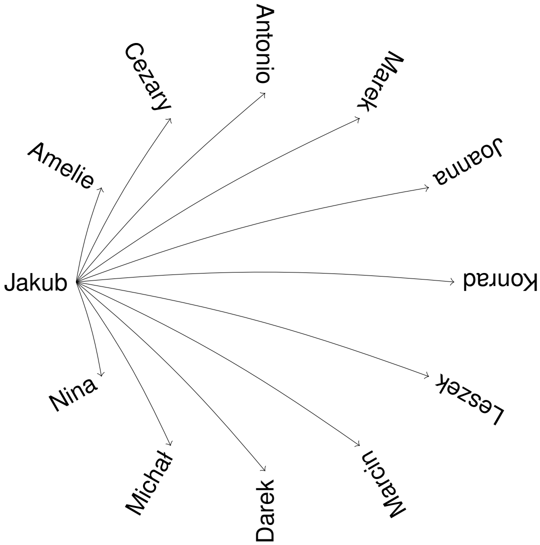

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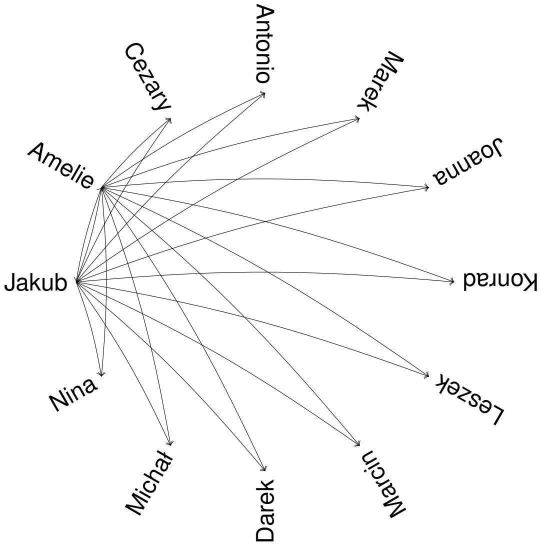

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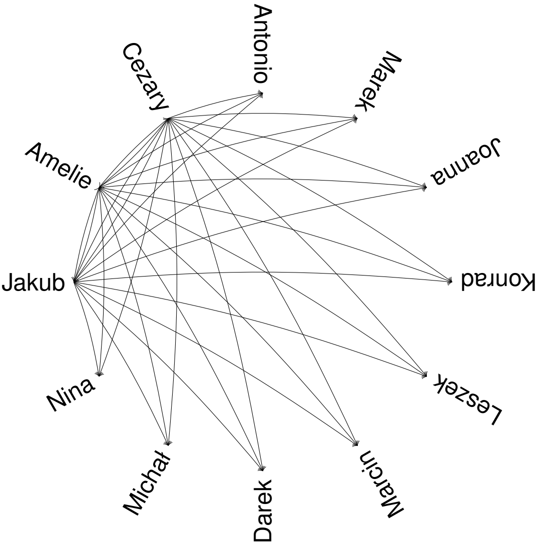

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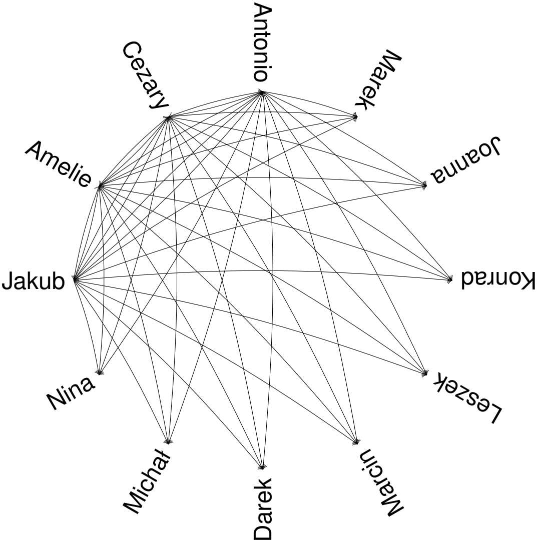

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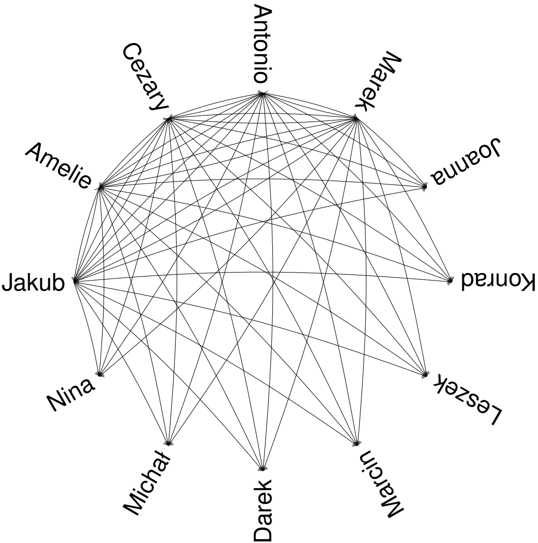

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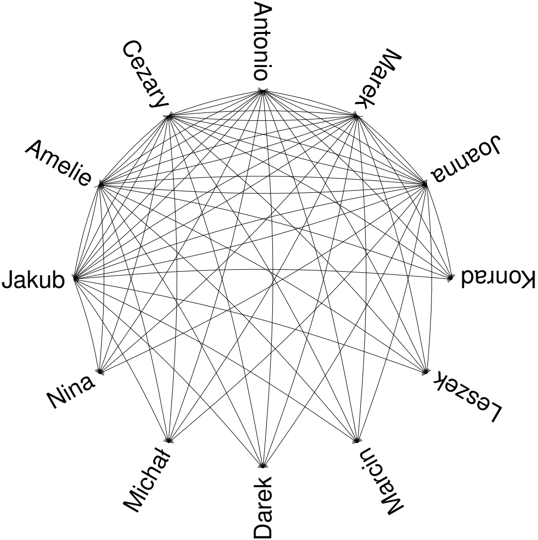

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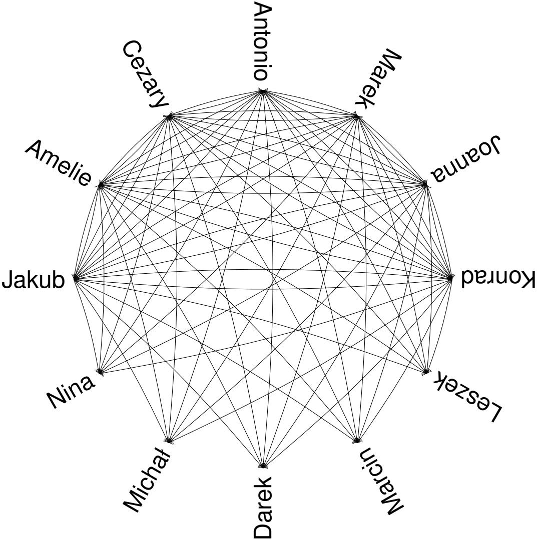

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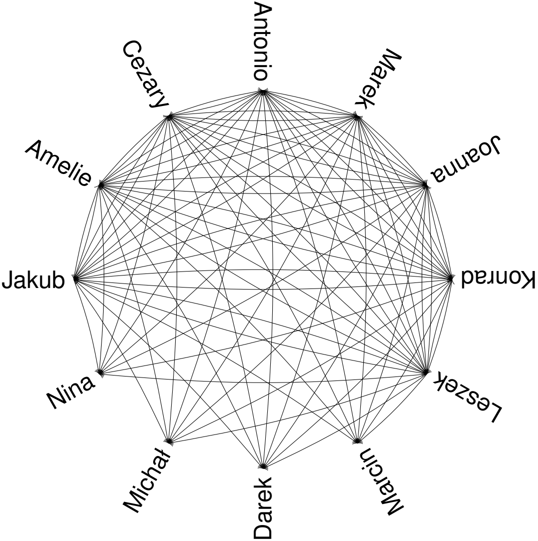

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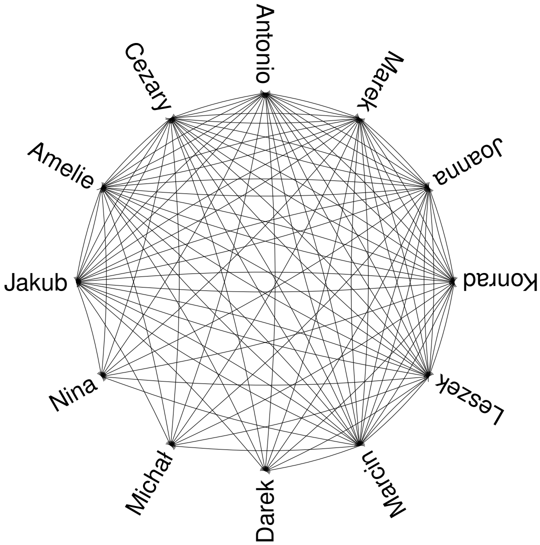

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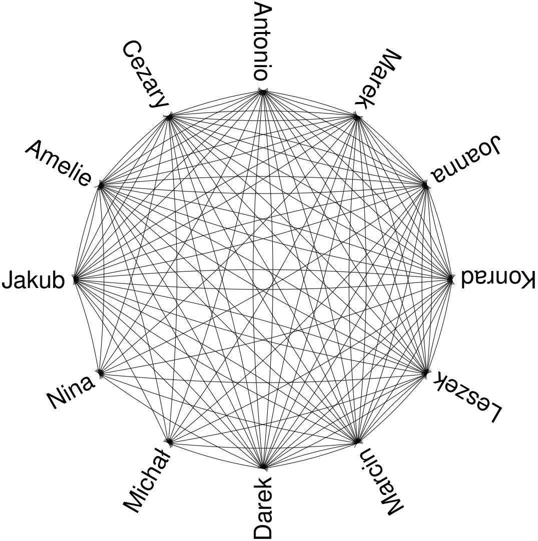

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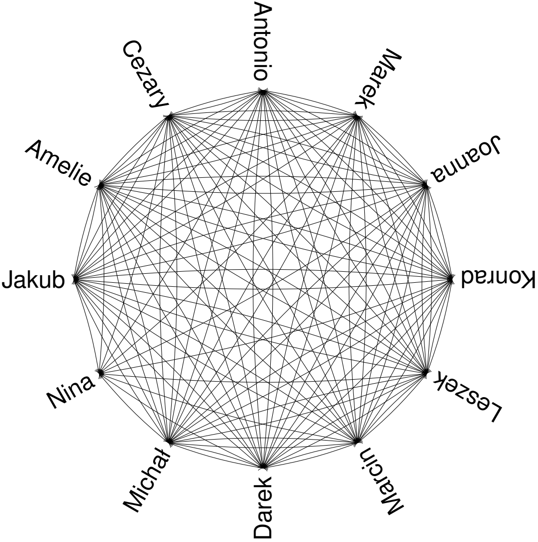

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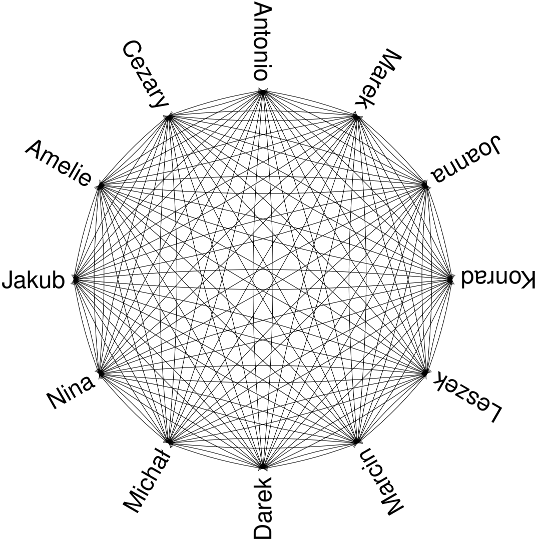


Multi-quantifier sentences in NL

1. Some relative of each villager and some relative of each townsman hate each other.
2. Most villagers and most townsmen hate each other.
3. Three PMs referred to each other indirectly.

$$\mathrm{Q}[A, B, R] \text{ or } \mathrm{Q}[A, R]$$

Problem

Fix an interpretation. How the complexity depends on Q*?*

Branching quantifiers

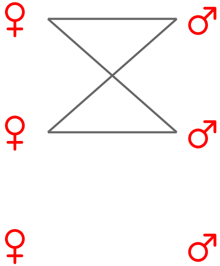
- Most girls and most boys hate each other.

$$\begin{array}{l} \text{most } x : G(x) \\ \text{most } y : B(y) \end{array} H(x, y).$$

$$\exists A \exists A' [\text{most}(G, A) \wedge \text{most}(B, A') \wedge \forall x \in A \, \forall y \in A' \;\, H(x, y)].$$

Illustration

- Most girls and most boys hate each other.



Branching readings are intractable

Theorem

Branching quantifiers are not FO-definable.

Theorem (Sevenster:2006)

Proportional branching sentences are NP-complete.

Observation (Gierasimczuk & Szymanik:2009)

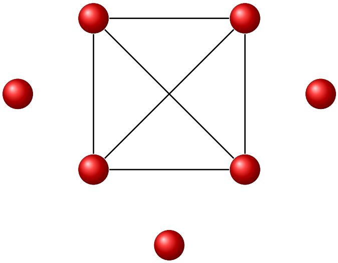
People tend to avoid branching interpretation.

Reciprocal sentences

1. Andi, Jarmo and Jakub laughed at one another.
2. 15 men are hitting one another.
3. Most of the PMs refer to each other.

Strong Meaning Hypothesis (Dalrymple et al. 1998)

- Most of the PMs refer to each other.



Strong reciprocal lift

Definition

Let Q be a right monotone increasing quantifier of type (1, 1). We define:

$$\mathrm{Ram}_{\mathrm{S}}(\mathrm{Q})[A, R] \iff \exists X \subseteq A[\mathrm{Q}(A, X) \\ \wedge \forall x, y \in X(x \neq y \implies R(x, y))].$$

Problem

For which quantifiers is it hard?

$\looparrowright$ It's interesting not only from a formal perspective but also as we know that it correlates with cognitive difficulty (Schlotterbeck & Bott:2013) and linguistic distributions (Szymanik & Thorne:2015), hence, it may factor into SMH.

Outlook

- Study some natural polynomial and NP-hard cases.
- Are all Ramseys either polynomial-time computable or NP-hard?
- No, there exist intermediate Ramsey quantifiers.
- Is there a natural characterization of the polynomial Ramseys?
- Yes, they are exactly the constant-log-bounded.

Outline

Generalized quantifiers

Definition (Lindström:1966)

Let $t = (n_1, \ldots, n_k)$ be a k-tuple of positive integers. *A generalized quantifier* of type t is a class Q of models of a vocabulary $\tau_t = \{R_1, \ldots, R_k\}$, such that R_i is n_i-ary for $1 \leq i \leq k$, and Q is closed under isomorphisms, i.e. if $\mathbb{M}$ and $\mathbb{M}'$ are isomorphic, then

$$(\mathbb{M} \in \mathsf{Q} \iff \mathbb{M}' \in \mathsf{Q}).$$

Example

$$\begin{aligned}
\text{every} &= \{(M, A, B) \mid A, B \subseteq M \text{ and } A \subseteq B\}.\\
\text{most} &= \{(M, A, B) \mid A, B \subseteq M \text{ and } \mathrm{card}(A \cap B) > \mathrm{card}(A - B)\}\\
\mathsf{W} &= \{(M, R) \mid R \subseteq M^2 \;\&\; R \text{ is a well-order}\}.\\
\text{Ram} &= \{(M, A, R) \mid A \subseteq M, R \subseteq M^2 \;\&\; \forall a, b \in A\; R(a, b)\}
\end{aligned}$$

Quantifiers in finite models

Finite models can be encoded as finite strings over some vocabulary.

Definition

By the *complexity of a quantifier* Q we mean the computational complexity of the corresponding class of finite models, that is, the complexity of deciding whether a given finite model belongs to this class.

$\looparrowright$ Think about sentence-picture verification or model-checking.

Outline

Basic proportional Ramsey quantifiers

Definition

For any rational number q between 0 and 1 we say that *the set* $A \subseteq U$ *is* q*-large relative to* U if and only if

$$\frac{card(A)}{card(U)} \geq q.$$

Definition

Let $\mathbb{M} = (M, S)$ be a relational model with universe M and one binary relation S. We say that $\mathbb{M} \in \mathrm{R_q}$ iff there is a q-large (relative to M) $A \subseteq M$ such that for all $a, b \in A$, $\mathbb{M} \models S(a, b)$.

Theorem

For every rational number q, such that $0 < q < 1$*, the corresponding Ramsey quantifier* $\mathrm{R_q}$ *is* NP*-complete.*

$\looparrowright$ Think about CLIQUE problems.

Arbitrary Ramsey quantifiers

Definition

We say that a *set* $A \subseteq U$ *is* f*-large relatively to* U iff

$$card(A) \geq f(card(U)).$$

Definition

We define $\mathrm{R_f}$ as the class of relational models $\mathbb{M} = (M, S)$, with universe M and a binary relation S, such that there is an f-large set $A \subseteq M$ such that for each $a, b \in A$, $\mathbb{M} \models S(a, b)$.

Corollary

Let $f(n) = \lceil rn \rceil$*, for some rational number* r *such that* $0 < r < 1$*. Then the quantifier* $\mathrm{R_f}$ *defines a* NP*-complete class of finite models.*

Bounded functions

Definition (Väänänen:1997)

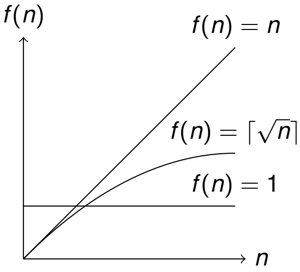
A function f *is bounded* if

$$\exists m \forall n [f(n) < m \vee n - m < f(n)].$$

Otherwise, f is *unbounded*.

Example

Typical bounded functions are: $f(n) = 1$ and $f(n) = n$. The first one is bounded from above by 2 as for every n we have $f(n) = 1 < 2$. The second one is bounded below by 1, for every n, $n - 1 < n$. Unbounded functions are for example: $\lceil \frac{n}{2} \rceil$, $\lceil \sqrt{n} \rceil$, $\lceil \log n \rceil$.



Logical intermezzo: bounded functions and definability

Boundness: $Q(X)$ iff there exists m such that X differs from the universe or empty set on at most m elements.

$$\exists X Q(X) \iff \forall t_1 \ldots \forall t_m \forall t_{m+1}$$
$$\Big[\Big(\bigwedge_{1 \leq i \leq m+1} X(t_i) \implies \bigvee_{1 \leq i < j \leq m+1} t_i = t_j \Big)$$
$$\vee \Big(\bigwedge_{1 \leq i \leq m+1} \neg X(t_i) \implies \bigvee_{1 \leq i < j \leq m+1} t_i = t_j \Big)\Big].$$

This formula says that X has a property Q if and only if X consists of at most m elements or X differs from the universe on at most m elements.

Tractable Ramsey quantifiers

Theorem

If f is polynomial-time computable and bounded, then the Ramsey quantifier R_f is polynomial-time computable.

Tractable Ramsey quantifiers

Theorem

If f is polynomial-time computable and bounded, then the Ramsey quantifier R_f *is polynomial-time computable.*

Problem

Does the Ramsey quantifier R_f *is either polynomial-time computable or* NP*-complete for all fs?*

Intermezzo: computational complexity

Problems in NP that are neither in P nor NP-complete are called NP-intermediate.

Theorem (Ladner:1975)

If P $\neq$ NP*, then* NPI *is not empty.*

Extra assumption: The Exponential Time Hypothesis

Exponential Time Hypothesis: *[Impagliazzo & Paturi:1999]*
3-SAT *cannot be solved in time* $2^{o(n)}$*,*
where n *denotes the number of variables in the input formula and, intuitively speaking, if a function* $f(n)$ *is* $o(g(n))$*, it means that* $g(n)$ *grows faster than* $f(n)$*, when the values for* n *get large enough.*

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A lower bound based on the ETH:

Theorem (Chen:2005)

*Assuming the ETH, there is no $m^{o(k)}$ time algorithm for k-*CLIQUE*, where m is the input size.*

Intermediate Ramsey quantifiers

Theorem

Let $f(n) = \lceil \log n \rceil$. The quantifier R_f is neither polynomial-time computable nor NP*-complete, unless the ETH fails.*

Intermediate Ramsey quantifiers

Theorem

Let $f(n) = \lceil \log n \rceil$. The quantifier $\mathrm{R_f}$ is neither polynomial-time computable nor NP-complete, unless the ETH fails.

Open Question

- *Is there an example from NL?*
- *Are there stronger assumptions for which $\mathrm{R_f}$ is either polynomial-time computable or NP-complete?*

Problem

Assuming ETH can we characterize polynomial Ramsey quantifiers?

Restricting class of functions

Observation

Let $f : \omega \longrightarrow \omega$ be a function that is not polynomial-time computable. Then $\mathrm{R_f}$ is not polynomial-time computable.

Assumption

The functions f that we consider are polynomial-time computable, i.e., for every $n \in \omega$, the value $f(n)$ is computable in time polynomial in n.

Constant-log-boundedness

Definition

Let $f : \omega \longrightarrow \omega$ be a computable function. We say that f is *constant-log-bounded* if one of the following holds:

- for all $n \in \omega$, $f(n)$ is bounded by a constant, i.e., there is some $m \in \omega$ such that for all $n \in \omega$ it holds that $f(n) \leq m$; or
- for all $n \in \omega$, $f(n)$ differs from n by at most $c \log n$, where c is some constant, i.e., there is some $c \in \omega$ such that for all $n \in \omega$ it holds that $f(n) \geq n - c \log n$.

Open Question

What is a logical (definability) or even linguistic interpretation?

Polynomial/non-polynomial dichotomy result

Theorem

Let $f : \omega \longrightarrow \omega$ be a computable function. Then, assuming the ETH, $\mathrm{R_f}$ is polynomial-time computable if and only if f is polynomial-time computable and constant-log-bounded.

Open Question

Can we get a trichotomy: isolating NP-hard from NPI?

Conclusions

- There are natural tractable and intractable Ramsey quantifiers.
- Under ETH, there exist intermediate Ramsey quantifier.
- Under ETH, we can characterize polynomial Ramsey quantifiers.

Open problems

Open Question

- *Is there a NL quantifier that is NPI?*
- *Which* $\mathrm{R_f}$*s enjoy stronger 'P vs NP-complete'-dichotomy?*
- *Is there a natural logical interpretation of constant-log-boundness?*
- *How to characterize full trichotomy?*
- *Is there a linguistic or cognitive interpretation of these borders?*

References

Ronald de Haan and Jakub Szymanik. A Dichotomy Result for Ramsey Quantifiers, Proceedings of the 22nd Workshop on Logic, Language, Information and Computation, V. de Paiva, R. de Queiroz, L.S. Moss, D. Leivant, A. G. de Oliveira (Eds.), 2015, pp. 69-80.


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