Computational complexity of some Ramsey quantifiers in finite models

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Outline

1. Introduction
   - Research motivation
   - Ramsey quantifiers
   - Computational complexity of quantifiers in finite models
   - INDEPENDENT SET and q–BIG CLIQUE
2. Ramsey quantifiers and their complexity

OUTLINE

Complexity and linguistic competence

1. Deciding whether some natural language sentence is true or not in a given finite situation.
2. Evaluating complexity of semantic construction is important for better understanding our linguistic competence.
3. Some natural language constructions are *NP*–complete - all known examples explore the idea of BQ.

Branching quantifiers and Ramsey quantifiers

1. *Some relative of each villager and some relative of each townsman hate each other.*
2. *Some book by every author is reffered to in some essay by every critic.*
3. *Most boys and most girls dated each other.*
4. All of the proofs of NP–completeness for BQ are based on some Ramsey property.

Finite Ramsey theorem and Ramsey quantifiers

Theorem

When coloring sufficiently large complete finite graph, one will find a big homogeneous subset, i. e. complete subgraph with all edges of the same colour of arbitrary large finite cardinality.

Definition

A Ramsey quantifier R is a generalized quantifier of the type (2), such that $M \models \mathrm{R}xy\ \varphi(x, y)$ exactly when there is $A \subseteq |M|$ (big relatively to the size of M) such that for each $a, b \in A$ $M \models \varphi(a, b)$.

Quantifiers and complexity

Definition

Let Q be of the type (n). By complexity of Q we mean the computational complexity of the class K_Q of finite models such that $M \in K_Q$ if and only if $M \models Qx_1, \ldots, x_n\, H(x_1, \ldots, x_n)$.

Definition

We say that Q is NP–hard if K_Q is NP–hard. Q is mighty if K_Q is NP and K_Q is NP–hard.

MIGHTY QUANTIFIERS - FIRST EXAMPLE

Let us consider models of the form $M = (U, E)$, where E is an equivalence relation.

DEFINITION

$M \models \mathrm{R_e} xy\, \varphi(x, y)$ means that there is a set $A \subseteq U$ such that $\forall a \in |M|\ \exists b \in A\ E(a, b)$ and for each $a, b \in A$ $M \models \varphi(a, b)$.

THEOREM (MOSTOWSKI, WOJTYNIAK 2004)

$\mathrm{R_e}$ *is mighty.*

MIGHTY QUANTIFIERS - SECOND EXAMPLE

Let us consider models of the form $M = (U, V, T)$, where V, T are subsets of U.

DEFINITION

$M \models$ BMost $xy\ \varphi(x, y)$ means that there are sets $A \subseteq U$ and $B \subseteq U$ such that MOST$x\ (V(x), A(x)) \wedge$ MOST$y\ (T(y), B(y)) \wedge \forall x \forall y (A(x) \wedge B(y) \Rightarrow H(x, y))$.

THEOREM (SEVENSTER 2006)

BMost *is mighty.*

INDEPENDENT SET

DEFINITION

Let $G = (V, E)$ be a graph and $I \subseteq V$. We say that I is independent if there is no $(i, j) \in E$ for any two $i, j \in I$.

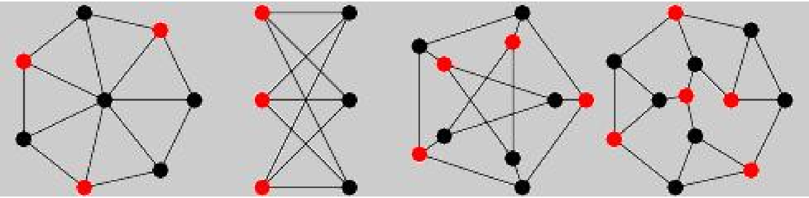


FIGURE: Independent sets

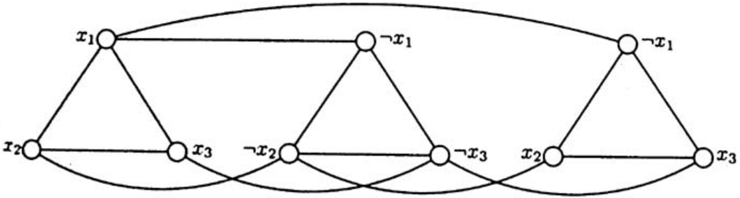
INDEPENDENT SET PROBLEM

Given graph $G = (V, E)$ and natural number k we must determine whether there is independent set in G of cardinality exactly k.

THEOREM

INDEPENDENT SET is NP–complete.

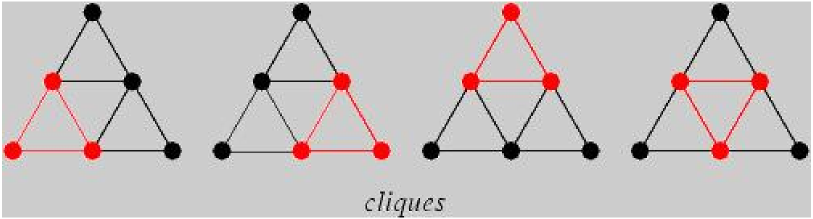
$$(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3)$$



CLIQUE PROBLEM

DEFINITION

We say that $A \subseteq V$ is a clique for graph $G = (V, E)$ if $A^2 \subseteq E$.
Problem: whether in G exists clique of cardinality k.



THEOREM

CLIQUE is NP–complete.

q–BIG CLIQUE PROBLEM

DEFINITION

We say that $A \subseteq V$ is q–big clique in $G = (V, E)$, if A is clique in G and $\frac{card(A)}{card(V)} \geq q$.

DEFINITION

Let $G = (V, E)$ and $q \in]0, 1[\cap \mathbb{Q}$. q–BIG CLIQUE problem is to decide if there is q–big clique $A \subseteq V$ in G.

q–BIG CLIQUE IS NP–COMPLETE

THEOREM

For $q = \frac{1}{3}$ q–BIG CLIQUE is NP–complete.

- We can consider graphs divided not only into disjoint triangles, but also complete quadrangles, pentagons, hexagons and so on ...

THEOREM

q–BIG CLIQUE is NP–complete for $q \geq \frac{1}{k}$, where $k > 2$.

q–BIG CLIQUE is NP–complete

Theorem

For every rational number $0 < q < 1$ q–BIG CLIQUE is NP–complete

Proof.

Let $G = (V, E)$ be such that $card(V) = ka$. We show that in G exists $\frac{1}{k}$–big clique iff in G' exists $\frac{m}{k}$–big clique for $m < k$, where $G' = (V', E')$ is constructed as follows:

- $V' = V \cup U$, where U such that $card(U) = n = \lceil \frac{(m-1)ka}{k-m} \rceil$ and $U \cap V = \emptyset$;
- $E' = E \cup U \times (U \cup V)$.

It suffices to observe that $\frac{n+a}{n+ka} \geq \frac{m}{k} > \frac{n+(a-1)}{n+ka}$. □

Ramsey quantifiers are NP–complete

Definition

Let $f : \mathbb{N} \longrightarrow \mathbb{N}$. A is f–big set when $card(A) \geq f(card(U))$.

Definition

$M \models \mathrm{R_f} xy\ \varphi(x, y)$ iff there is f–big $A \subseteq |M|$ such that for each $a, b \in A$, $M \models \varphi(a, b)$.

Theorem

Let $f_r(n)$ be the integer part of rn, for some rational r such that $0 < r < 1$. Then $\mathrm{R_{f_r}}$ is mighty.

Future work

Conjecture

Let $k > 2$ and $0 < m < k$. There is PTIME class of graphs J and NP–complete class $K \subseteq J$ s. t. for any $G \in J$ we have:

- *$G \in K$ iff there is a clique in G of size $\geq \frac{m}{k} \times card(G)$;*
- *$G \notin K$ iff there is no clique in G of size $> \frac{m-1}{k} \times card(G)$.*

Conclusion

Let f be such that $\lim_{n \to \infty} \frac{f(n)}{n} = a$ exists and $0 < a < 1$. Then $\mathrm{R_f}$ is NP–hard.

Conclusion

If f satisfies the assumptions of the previous theorem and f is PTIME computable, then $\mathrm{R_f}$ is mighty.

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