Logic for A.I. - Solutions

Rosalie Iemhoff

October 23, 2008

1 Solutions Exercises 1

Ex. 17

Let us call the new system L, i.e. its axioms are all propositional tautologies (Axiom 1) plus the axioms $\Box\top$ (Axiom 2) and $\Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi)$ (Axiom 3), and the rules modus ponens and

$$\frac{\phi \to \psi}{\Box\phi \to \Box\psi}$$

We have to show that for all formulas ϕ

$$\vdash_{\mathsf{K}} \phi \ \Leftrightarrow \vdash_{\mathsf{L}} \phi.$$

$\Rightarrow$: For this direction we have to show that L derives all axioms of K and all its rules.

Axiom 1 of K is the same as Axiom 1 in L, thus we have nothing to prove.

Axiom 2 of K is $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$. We give a derivation of this formula in L:

$$\dfrac{\dfrac{\dfrac{(\phi \to \psi) \wedge \phi \to \psi}{\Box((\phi \to \psi) \wedge \phi) \to \Box\psi}\ \text{(the rule from L)}}{\Box(\phi \to \psi) \wedge \Box\phi \to \Box\psi}\ \text{(axiom 3 of L and propositional logic)}}{\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)}\ \text{(propositional logic)}$$

Remain the rules of K. Modus ponens is a rule of both so there is nothing to prove. We show that L proves the Necessitation rule. That is, we have to show that if $\vdash_{\mathsf{L}} \phi$, then $\vdash_{\mathsf{L}} \Box\phi$. The following derivation in L from assumption ϕ shows this:

$$\dfrac{\dfrac{\dfrac{\phi}{\top \to \phi}\ \text{(propositional logic)}}{\Box\top \to \Box\phi}\ \text{(the rule from L)}}{\Box\phi}\ \text{(modus ponens using axiom } \Box\top)$$

This completes the direction of the proof form left to right.

$\Leftarrow$: For this direction we have to show that K derives all axioms of L and all its rules.

Axiom 1 of L is the same as Axiom 1 in K, thus we have nothing to prove.

Axiom 2 of L is $\Box\top$. But since we have $\vdash_{\mathsf{K}} \top$ ($\top$ clearly being a tautology), we also have $\vdash_{\mathsf{K}} \Box\top$ by Necessitation.

Axiom 3 of L is $\Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi)$. That K derives this formula is shown as follows:

$$\dfrac{\dfrac{\dfrac{\phi \to (\psi \to (\phi \wedge \psi))}{\Box\big(\phi \to (\psi \to (\phi \wedge \psi))\big)}\ \text{(necessitation rule)}}{\Box\phi \to (\Box\psi \to \Box(\phi \wedge \psi))}\ \text{(two applications of axiom 2 of K plus propositional logic)}}{\Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi)}\ \text{(propositional logic)}$$

Remain the rules of L. Modus ponens is a rule of both so there is nothing to prove. We show that K proves the other rule of L. That is, we have to show that if $\vdash_{\mathsf{K}} \phi \to \psi$, then $\vdash_{\mathsf{K}} \Box\phi \to \Box\psi$. The following derivation in K from assumption $\phi \to \psi$ shows this:

$$\dfrac{\dfrac{\phi \to \psi}{\Box(\phi \to \psi)}\ \text{(necessitation rule)}}{\Box\phi \to \Box\psi}\ \text{(axiom 2 of K plus propositional logic)}$$

This completes the direction of the proof from right to left. And thus we have proved the exercise.

Ex. 19

Let us first consider some examples. Indeed, if $\phi \to \psi$ is derivable in K, then so is $\Box\phi \to \Box\psi$ by necessitation and Axiom 2 of K. But then also the formula $\neg\Box\psi \to \neg\Box\phi$ is derivable by contraposition. And then, again by necessitation and Axiom 2, also $\Box\neg\Box\psi \to \Box\neg\Box\phi$ is derivable. By contraposition thus also $\neg\Box\neg\Box\phi \to \neg\Box\neg\Box\psi$. (Contraposition means: if we prove $\varphi \to \chi$, then also $\neg\chi \to \neg\varphi$ holds, as it is a formula that is equivalent to the former formula.)

Thus if $\phi \to \psi$ is derivable, then so are $M\phi \to M\psi$ for $M = \Box$ and $M = \neg\Box\neg\Box$, and $M\psi \to M\phi$ for $M = \neg\Box$ and $\Box\neg\Box$. Note that in the former case the number of negations in M is even, and in the latter case it is odd.

The general case we prove by induction on the number of symbols, n, in M. Suppose $\phi \to \psi$ is derivable in K. We have to show that $M\phi \to M\psi$ is derivable in case the number of negations in M is even, and $M\psi \to M\phi$ is derivable in case the number of negations in M is odd.

(Base case $n = 0$) In this case M is an empty sequence. Thus $M\phi \to M\psi$ is equal to $\phi \to \psi$, and thus it follows that $M\phi \to M\psi$ is derivable.
(Case $n + 1$) In this case $M = \Box N$ or $M = \neg N$ for a sequence N of boxes and negations that contains n symbols. We split this case in four separate cases:
Case $M = \Box N$ and the number of negations in N is even. Note that in this

case the number of occurrences of negations in M is the same as in N, and thus even. By the induction hypothesis on N we have that $N\phi \to N\psi$ is derivable. But then so is $\Box N\phi \to \Box N\psi$ by necessitation and Axiom 2. Thus $M\phi \to M\psi$ is derivable.
Case $M = \Box N$ and the number of negations in N is odd. Note that in this case the number of occurrences of negations in M is the same as in N, and thus odd. By the induction hypothesis on N we have that $N\psi \to N\phi$ is derivable. But then so is $\Box N\psi \to \Box N\phi$ by necessitation and Axiom 2. Thus $M\psi \to M\phi$ is derivable.
Case $M = \neg N$ and the number of negations in N is even. Note that in this case the number of occurrences of negations in M is one more that the number of negations in N, and thus odd. Hence we have to show that $M\psi \to M\phi$ is derivable. By the induction hypothesis on N we have that $N\phi \to N\psi$ is derivable. But then so is $\neg N\psi \to \neg N\phi$ by contraposition. That is, $M\psi \to M\phi$ is derivable.
Case $M = \neg N$ and the number of negations in N is odd. Note that in this case the number of negations in M is one more that the number of negations in N, and thus even. Hence we have to show that $M\phi \to M\psi$ is derivable. By the induction hypothesis on N we have that $N\psi \to N\phi$ is derivable. But then so is $\neg N\phi \to \neg N\psi$ by contraposition. That is, $M\phi \to M\psi$ is derivable. This completes the proof.

Ex. 27

For all frames F:

$$F \models \Box\bot \text{ if and only if } F \text{ is completely disconnected.}$$

Proof $\Leftarrow$: Suppose $F = (W, R)$ is completely disconnected, i.e. $\neg(wRv)$ for all worlds w and v in W. We have to show that $F \models \Box\bot$, that is, that for all valuations V, for all $w \in W$, $w \models \Box\bot$ in the model (W, R, V). Thus consider an arbitrary valuation V and an arbitrary world w in W. Since F is completely disconnected w has no successors. $w \models \Box\bot$ means that $v \models \bot$ for all successors of w. But since w has no successors, this is trivially true. Thus $w \models \Box\bot$ indeed. Note that we have in fact shown that in this model $w \models \Box\phi$ for all ϕ.
$\Rightarrow$: This direction we show by contraposition. Thus we assume $F = (W, R)$ is not completely disconnected, and then show that $F \not\models \Box\bot$. In other words, we have to show that if F is not completely disconnected, then there is a valuation V and a world w in W such that $w \not\models \Box\bot$ in the model (W, R, V). Thus suppose F is not completely disconnected. Then there are at least two worlds w and v (possibly the same) such that wRv. Observe that $w \models \Box\bot$ means that all successors of w, thus in particular v, force $\bot$. Since in every model never $v \models \bot$, it follows that $w \not\models \Box\bot$. □

Ex. 28

$\Box\Box\bot$.

Ex. 30

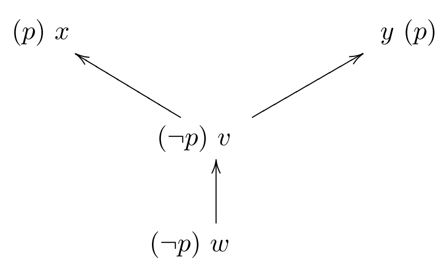
For all frames F:

$$F \models \Diamond\Box\varphi \to \varphi \text{ if and only if } F \text{ is symmetric.}$$

Proof $\Leftarrow$: Suppose $F = (W, R)$ is symmetric. We have to show that $F \models \Diamond\Box\varphi \to \varphi$, that is, that for all formulas φ, for all valuations V, for all $w \in W$, $w \models \Diamond\Box\varphi \to \varphi$ in the model (W, R, V). Thus consider an arbitrary formula φ, an arbitrary valuation V and an arbitrary world w in W. Now suppose $w \models \Diamond\Box\varphi$. We have to show that $w \models \varphi$. $w \models \Diamond\Box\varphi$ implies the existence of a world v such that wRv and $v \models \Box\varphi$. But the symmetry of F implies vRw. And thus $w \models \varphi$, since $v \models \Box\varphi$.
$\Rightarrow$: This direction we show by contraposition. Thus we assume $F = (W, R)$ is not symmetric, and then show that $F \not\models \Diamond\Box\varphi \to \varphi$. In other words, we have to show that if F is not symmetric, then there is a formula φ and a valuation V and a world w in W such that $w \not\models \Diamond\Box\varphi \to \varphi$ in the model (W, R, V). Note that $w \not\models \Diamond\Box\varphi \to \varphi$ is equivalent to $w \models \Diamond\Box\varphi \wedge \neg\varphi$. Thus suppose F is not symmetric. Then there are at least two worlds w and v such that wRv and not vRw. Now define the valuation V on F as follows:

$$u \in V(p) \Leftrightarrow vRu.$$

Thus, we put $u \models p$ if vRu, and for all other nodes u in W we put $u \models \neg p$. E.g. as in this model:



Since not vRw, we have $w \models \neg p$. Also, $v \models \Box p$ follows from the definition of V. Since wRv this implies that $w \models \Diamond\Box p$. And thus $w \models \Diamond\Box p \wedge \neg p$. Hence we have a found a formula φ, namely p, for which $w \models \Diamond\Box\varphi \wedge \neg\varphi$, and this completes the proof. $\Box$

Ex. 31

$\Box^n\bot$ ($\Box^n\bot$ is $\Box\Box\ldots\Box\bot$, n times $\Box$).

Ex. 34

Recall that irreflexive is $\forall w \neg wRw$, and thus it is not the same as not reflexive, which is $\exists w \neg wRw$. Suppose R is well-founded. We show that it is irreflexive and asymmetric.

If there would be a node w such that wRw, then clearly there would be $w_1, w_2, \ldots$ such that $\ldots w_3 R w_2 R w_1 R w$, because we can take $w_n = w$ for all $n \geq 1$. Similarly, if R would not be asymmetric, there would be two nodes w and v such that wRv and vRw. Thus we can find a chain $\ldots w_3 R w_2 R w_1 R w$, by taking $w_{2n+1} = v$ and $w_{2n} = w$, for all $n \geq 0$.

Ex. 37

Recall from the notes that if a set has a model, then it is consistent. Thus it suffices to provide a model for the given set. We leave the construction of such a model to you.

The set is not consistent in T, as e.g. $\Box\Box p$ implies $\Box p$ in this system, and thus the set would then derive $\Box p \wedge \neg \Box p$.

Ex. 41

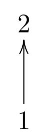
The bisimulation Z is:

$$Z = \{\langle w, 1\rangle, \langle x, 2\rangle, \langle y, 2\rangle, \langle z, 2\rangle, \langle v, 2\rangle\}.$$

Less formal, one may also say: Z satisfies exactly $wZ1$ and $xZ2$, $yZ2$, $zZ2$ and $vZ2$. It is instructive to check for yourself that Z is indeed a bisimulation.

Ex. 43

F_{v_1} validates $\Box\bot$, which F does not. The frame



is a p-morphic image of F. The p-morphism f is:

$$f = \{\langle w, 1\rangle\} \cup \{\langle v_i, 2\rangle \mid i = 1, 2, \ldots\}.$$

Less formal, one may also say: $f(w) = 1$ and $f(v_i) = 2$ for all $i \geq 1$.

Ex. 44

$\Box^n \bot$ does not hold in the frame. E.g., it is not forced in any v_m with $m > n$. But there are many more nodes in which it is not forced: w, many u_i, many x_i, etc. $\Diamond\top$ does not hold either, as it does not hold in e.g. v_1.

We show that no finite frame can be the p-morphic image of F, by showing that if f is a p-morphism from F to a frame G, then G has to be infinite. We know that for the models M and N on F and G in which we do not force any propositional variables, $w \models_M \varphi \;\Leftrightarrow\; w \models_N \varphi$ holds, by the p-morphism theorem. Now note that for all $i \geq 1$, $v_i \models \neg\Box\bot \wedge \neg\Box^2\bot \wedge \ldots \wedge \neg\Box^{i-1}\bot \wedge \Box^i\bot$. Thus all these v_i have to be mapped to different nodes under f, from which it follows that G is infinite.

Ex. 45

If there would be a p-morphism f, then y should be mapped to either 1 or 2. We show that both cases cannot occur. Call the accessibility relation in the left frame R and in the right frame R'. If $f(y) = 1$, then because $1R'2$, there should be a node a such that yRa and $f(a) = 2$. But there is no such node. If $f(y) = 2$, then because $2R'2$, there should be a node a such that yRa and $f(a) = 2$. But there is no such node.

Are there models on . . . ? No, $w \models \Diamond\Box\bot$ and $1 \not\models \Diamond\Box\bot$.

Is the generated . . . ? Yes.

Ex. 49

The formula is the conjunction of the formula that characterizes the reflexive and the formula that characterizes the transitive frames: $(\Box\varphi \to \varphi) \wedge (\Box\varphi \to \Box\Box\varphi)$. We show that indeed

$$F \models (\Box\varphi \to \varphi) \wedge (\Box\varphi \to \Box\Box\varphi) \;\Leftrightarrow\; F \text{ is a reflexive transitive frame.}$$

We use the characterization theorem for the reflexive and for the transitive frames treated in the notes.

$\Rightarrow$: Suppose that $F \models (\Box\varphi \to \varphi) \wedge (\Box\varphi \to \Box\Box\varphi)$. Hence $F \models \Box\varphi \to \varphi$ and $F \models \Box\varphi \to \Box\Box\varphi$. By the mentioned theorem it follows that F is transitive and reflexive.

$\Leftarrow$: Suppose F is transitive and reflexive. By the mentioned theorem it follows that $F \models \Box\varphi \to \varphi$ and $F \models \Box\varphi \to \Box\Box\varphi$. Hence $F \models (\Box\varphi \to \varphi) \wedge (\Box\varphi \to \Box\Box\varphi)$.

Ex. 50

Theorem 1 (Valuation theorem) For any maximal K-consistent set of formulas Γ (that is, for any node in the canonical model), for any formula φ:

$$\Gamma \models \varphi \;\Leftrightarrow\; \varphi \in \Gamma.$$

(Note that here $\Gamma \models \varphi$ means that Γ forces φ in the canonical model.)

Proof We prove the statement by formula induction on φ. Thus we show that it holds for atomic formulas, and then, assuming it holds for ϕ and ψ, we show that it holds for $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \to \psi$, $\neg\phi$ and $\Box\phi$.

Suppose φ is a propositional variable p. Then $\Gamma \models p \iff p \in \Gamma$ by the definition of the canonical model.

Suppose $\varphi = \phi \wedge \psi$ and that $\Pi \models \phi \iff \phi \in \Pi$ and $\Pi \models \psi \iff \psi \in \Pi$ have already been proved for any Π (the induction hypothesis), in particular for Γ. We show that

$$\Gamma \models \phi \wedge \psi \iff \phi \wedge \psi \in \Gamma.$$

$\Rightarrow$: Suppose $\Gamma \models \phi \wedge \psi$. Then $\Gamma \models \phi$ and $\Gamma \models \psi$. By the induction hypothesis $\phi \in \Gamma$ and $\psi \in \Gamma$. Because Γ is maximal consistent, either $\phi \wedge \psi \in \Gamma$ or $\neg(\phi \wedge \psi) \in \Gamma$. We show that the last case cannot occur: since $\phi \in \Gamma$ and $\psi \in \Gamma$, $\Gamma \vdash_{\mathsf{K}} \phi \wedge \psi$. Hence $\neg(\phi \wedge \psi) \in \Gamma$ would imply $\Gamma \vdash_{\mathsf{K}} (\phi \wedge \psi) \wedge \neg(\phi \wedge \psi)$, which cannot be because Γ is consistent. Hence $\phi \wedge \psi \in \Gamma$, and we are done.

$\Leftarrow$: Suppose $\phi \wedge \psi$ is in Γ. We show that $\phi, \psi \in \Gamma$. First $\phi \in \Gamma$. Because Γ is maximal consistent, either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$. Suppose $\neg\phi \in \Gamma$. We show that this cannot be the case. Then $\phi \in \Gamma$ will follow. For suppose $\neg\phi \in \Gamma$. Then Γ would derive $\neg(\phi \wedge \psi)$ and $\phi \wedge \psi$, since $\neg\phi \vdash_{\mathsf{K}} \neg(\phi \wedge \psi)$, and thus $\phi \wedge \psi, \neg\phi \vdash_{\mathsf{K}} \neg(\phi \wedge \psi) \wedge (\phi \wedge \psi)$, and thus $\Gamma \vdash_{\mathsf{K}} \neg(\phi \wedge \psi) \wedge (\phi \wedge \psi)$. But this cannot be, because Γ is consistent. Thus $\neg\phi$ cannot be in Γ, and whence $\phi \in \Gamma$. The same argument applies to ψ. Thus we have shown that $\phi \in \Gamma$ and $\psi \in \Gamma$. Now the induction hypothesis implies that $\Gamma \models \phi$ and $\Gamma \models \psi$. But then $\Gamma \models \phi \wedge \psi$ follows, and we are done.

The cases for the other connectives are similar.

The last case, suppose $\varphi = \Box\phi$ and that $\Pi \models \phi \iff \phi \in \Pi$ has already been proved for all Π (the induction hypothesis). We show that

$$\Gamma \models \Box\phi \iff \Box\phi \in \Gamma.$$

$\Rightarrow$: Suppose $\Gamma \models \Box\phi$. We have to show that $\Box\phi \in \Gamma$. Because Γ is maximal consistent either $\Box\phi$ or $\neg\Box\phi$ is an element of Γ. We show that the last case cannot occur. For if $\neg\Box\phi \in \Gamma$, there is a maximal consistent set Π such that $\neg\phi \in \Pi$ and for all $\Box\chi \in \Gamma$, $\chi \in \Pi$, i.e. $\Gamma R_{\mathsf{K}} \Pi$. To see that such a Π exists requires a somewhat longer argument, and we leave it unproved, and just state the fact here. Since $\neg\phi \in \Pi$, then also $\Pi \not\models \phi$ by the induction hypothesis, and thus $\Gamma \not\models \Box\phi$, which we assumed. This shows that $\neg\Box\phi \in \Gamma$ cannot be the case, and thus $\Box\phi \in \Gamma$ follows, and that is what we had to show.

$\Leftarrow$: Suppose $\Box\phi \in \Gamma$. We have to show that $\Gamma \models \phi$, i.e. for all Π with $\Gamma R_{\mathsf{K}} \Pi$, $\Pi \models \phi$ holds. Suppose $\Gamma R_{\mathsf{K}} \Pi$. By the definition of R_{K} it follows that $\phi \in \Pi$. By the induction hypothesis it follows that $\Pi \models \phi$. This shows that $\Gamma \models \Box\phi$.

□

Ex. 51

Theorem 2 If there is a bisimulation Z between two models $M = (W, R, V)$ and $M' = (W', R', V')$, then for all $w \in W$ and $w' \in W'$ such that wZw' holds, for all formulas φ:

$$w \models_M \varphi \;\Leftrightarrow\; w' \models_{M'} \varphi.$$

Proof Suppose Z, M, M' are as in the theorem and consider $w \in W$ and $w' \in W'$ such that wZw'. We leave out the subscripts M and M' at $\models$, as it is clear which models are meant. We prove the statement by formula induction on φ. Thus we show that it holds for atomic formulas, and then, assuming it holds for ϕ and ψ, we show that it holds for $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \to \psi$, $\neg\phi$ and $\Box\phi$.

Suppose φ is a propositional variable p. Then $w \models p \;\Leftrightarrow\; w' \models p$ follows from the definition of bisimulation.

Suppose $\varphi = \phi \wedge \psi$ and that

$$v \models \phi \;\Leftrightarrow\; v' \models \phi \qquad v \models \psi \;\Leftrightarrow\; v' \models \psi$$

has already been proved for all $v \in W$ and $v' \in W'$ such that vZv' (the induction hypothesis). We have to show that

$$w \models \phi \wedge \psi \;\Leftrightarrow\; w' \models \phi \wedge \psi.$$

$\Rightarrow$: suppose $w \models \phi \wedge \psi$. Thus $w \models \phi$ and $w \models \psi$. By the induction hypothesis $w' \models \phi$ and $w' \models \psi$. Hence $w' \models \phi \wedge \psi$, and that is what we had to prove. The case $\Leftarrow$ is completely similar.

The cases $\phi \vee \psi$, $\phi \to \psi$, $\neg\phi$ have a similar argument. We only treat $\Box\phi$. So, assuming that

$$v \models \phi \;\Leftrightarrow\; v' \models \phi$$

has already been proved for all $v \in W$ and all $v' \in W'$ such that vZv' (induction hypothesis), we show that

$$w \models \Box\phi \;\Leftrightarrow\; w' \models \Box\phi$$

holds.

$\Rightarrow$: we show this by contraposition. Thus we assume $w' \models \neg\Box\phi$, and show that then $w \models \neg\Box\phi$ will follow. Thus assume $w' \models \neg\Box\phi$. By the definition of forcing there has to be a $v' \in W'$ such that $w'R'v'$ and $v' \not\models \phi$. By the definition of bisimulation there is a $v \in W$ such that wRv and vZv'. By the induction hypothesis on v and ϕ it follows that $v \not\models \phi$. But since wRv, then $w \models \neg\Box\phi$ follows, and that is what we had to show. The direction $\Leftarrow$ is completely similar. This completes the proof. $\Box$

Ex. 54

We start with T and show that the frame of the T-canonical model is reflexive. Let R be the relation of the T-canonical model. Recall that for two maximal T-consistent sets Γ and Π

$$\Gamma R \Pi \;\Leftrightarrow\; \forall\varphi(\Box\varphi \in \Gamma \;\Rightarrow\; \varphi \in \Pi).$$

Thus to show that R is reflexive we have to show that

$$\forall\varphi(\Box\varphi \in \Gamma \;\Rightarrow\; \varphi \in \Gamma).$$

But this follows from the axiom $\Box\varphi \to \varphi$. For suppose $\Box\varphi \in \Gamma$. Because Γ is maximal T-consistent $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$. $\neg\varphi$ cannot be in Γ, since the axiom $\Box\varphi \to \varphi$ then would imply that $\Gamma \vdash_{\mathsf{T}} \varphi \wedge \neg\varphi$, which cannot be as Γ is consistent. Thus $\varphi \in \Gamma$, and this is what we had to show.

The case $\mathsf{K4}$. We have to show that the frame of the $\mathsf{K4}$-canonical model is transitive. Let R be the relation of the $\mathsf{K4}$-canonical model. Thus we have to show that $\Gamma R \Pi R \Theta$ implies $\Gamma R \Theta$. Using the definition of R on canonical models, This means that we have to show that $\Gamma R \Pi R \Theta$ implies

$$\forall\varphi(\Box\varphi \in \Gamma \;\Rightarrow\; \varphi \in \Theta).$$

But this follows from the axiom $\Box\varphi \to \Box\Box\varphi$. For suppose $\Box\varphi \in \Gamma$. Because Γ is maximal $\mathsf{K4}$-consistent $\Box\Box\varphi \in \Gamma$ or $\neg\Box\Box\varphi \in \Gamma$. $\neg\Box\Box\varphi$ cannot be in Γ, since the axiom $\Box\varphi \to \Box\Box\varphi$ then would imply that $\Gamma \vdash_{\mathsf{K4}} \Box\Box\varphi \wedge \neg\Box\Box\varphi$, which cannot be as Γ is consistent. Thus $\Box\Box\varphi \in \Gamma$. Hence $\Box\varphi \in \Pi$ since $\Gamma R \Pi$. But then $\varphi \in \Theta$ since $\Pi R \Theta$, and this is what we had to show.

Ex. 55

$K_a\phi \to K_b\phi$, $K_cK_a\varphi \to K_bK_a\varphi$, $K_a\psi \to \neg K_b\neg\psi$, $K_aK_bK_a\varphi$.

Ex. 56

$$F \models \Box_a\varphi \to \Box_b\varphi \;\Leftrightarrow\; R_b \subseteq R_a.$$

$\Leftarrow$: Suppose $R_b \subseteq R_a$, and that $w \models \Box_a\varphi$ in a model on the frame. We have to show that $w \models \Box_b\varphi$, i.e. $\forall v(wR_bv \Rightarrow v \models \varphi)$. If wR_bv, then wR_av because $R_b \subseteq R_a$. Thus $v \models \varphi$ since $w \models \Box_a\varphi$, and that is what we had to show.

$\Rightarrow$: this we prove by contraposition. Suppose $R_b \not\subseteq R_a$, i.e. there are wR_bv such that not wR_av. Define

$$x \in V(p) \;\Leftrightarrow\; wR_ax.$$

We leave it to you to check that indeed $w \models \Box_ap$, and $w \models \neg\Box_bp$. This then shows that not for all φ, $\Box_a\varphi \to \Box_b\varphi$ holds on the frame.

Ex. 58

$$F \models \Diamond_1\varphi \to \Diamond_2\Diamond_2\varphi \;\Leftrightarrow\; \forall w\forall v(wR_1v \Rightarrow \exists u(wR_2uR_1v)).$$

$\Leftarrow$: Suppose $\forall w\forall v(wR_1v \Rightarrow \exists u(wR_2uR_1v))$, and that $w \models \Diamond_1\varphi$ in a model on the frame, i.e. there exists a v such that wR_1v and $v \models \varphi$. We have to show that $w \models \Diamond_2\Diamond_1\varphi$, i.e. $\exists u\exists z(wR_2uR_1z \wedge z \models \varphi)$. But if wR_1v, then wR_2uR_1v for some u, because of the property of the frame. Thus we can take $z = v$, and then indeed have that $(wR_2uR_1z \wedge z \models \varphi)$, and that is what we had to show.

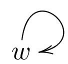
⇒: this we prove by contraposition. Suppose that $\forall w \forall v(wRv \Rightarrow \exists u(wR_2uR_1v))$ does not hold, i.e. there are w and v such that wR_1v and for no u wR_2uR_1v holds. Define

$$V(x,p) = 1 \text{ if } x = v, \text{ and } V(p,x) = 0 \text{ otherwise.}$$

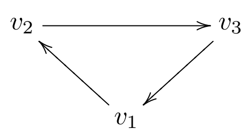
Note that thus p is only forced at node v. Indeed, $w \models \Diamond_1 p$, as wR_1v and $v \models p$. But $w \models \neg\Diamond_2\Diamond_1 p$, since v is the only node that forces p, and thus for $w \models \Diamond_2\Diamond_1 p$ to hold, there should be a u such that wR_2uR_1v, but there is no such u. This shows that not for all φ, $\Diamond_1\varphi \to \Diamond_2\Diamond_1\varphi$ holds on the frame.

Ex. 59

Consider the frame F



Let G be the frame



It is not difficult to see that F is a p-morphic image of G. But G is asymmetric and F is not (it is symmetric). If the class of asymmetric frames were characterized by a formula ϕ, then it would follow that $G \models \phi$ and $F \not\models \phi$, which contradicts th eP-mophism theorem.