Homework #3

November 27, 2009

The assignments are due Friday, December 4 at 11AM in class.

Exercise 1 (3pt) Prove the Lindenbaum's Lemma: If Σ is a K-consistent set of formulas then there is a K-maximal consistent set of formulas Σ^+ such that $\Sigma \subseteq \Sigma^+$.

Proof Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be an enumeration of the formulas of our language. We define the set Σ^+ as the union of a chain of K-consistent sets as follows:

$$\Sigma_0 = \Sigma$$

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{\varphi_n\} & \text{if this is K-consistent} \\ \Sigma_n \cup \{\neg\varphi_n\} & \text{otherwise} \end{cases}$$

$$\Sigma^+ = \bigcup_{n \geq 0} \Sigma_n$$

To finish the proof you need to show the following properties:

1. $\Sigma \subseteq \Sigma^+$
2. Σ^+ is K-consitent
3. Σ^+ is maximal

$\square$

Exercise 2 (5pt) Prove the valuation lemma: For any maximal K-consistent set of formulas Γ (that is, for any node in the canonical model), for any formula φ:

$$M_K, \Gamma \models \varphi \iff \varphi \in \Gamma.$$

Hint: induction on φ.

Exercise 3 (2pt) Show that the following rule is admissible in K: if $\vdash_K \Box\alpha \vee \Box\beta$, then $\vdash_K \alpha$ or $\vdash_K \beta$. (Hint: K is complete)