Introduction to Generalized Quantifier Theory

Jakub Szymanik

Institute for Logic, Language and Computation
University of Amsterdam

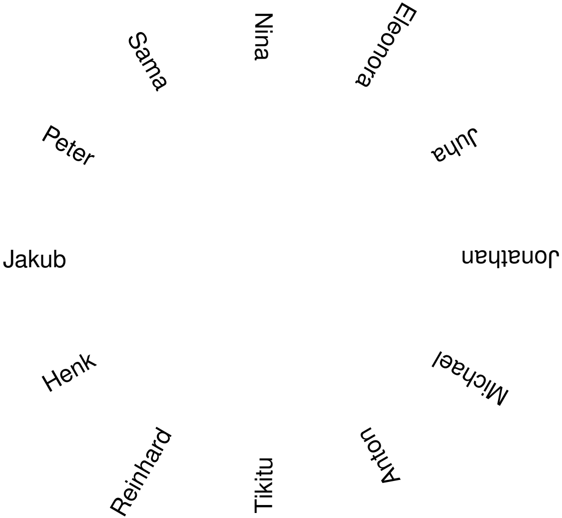
EGG 2013

Quantifiers are useful

Everyone knows everyone here.

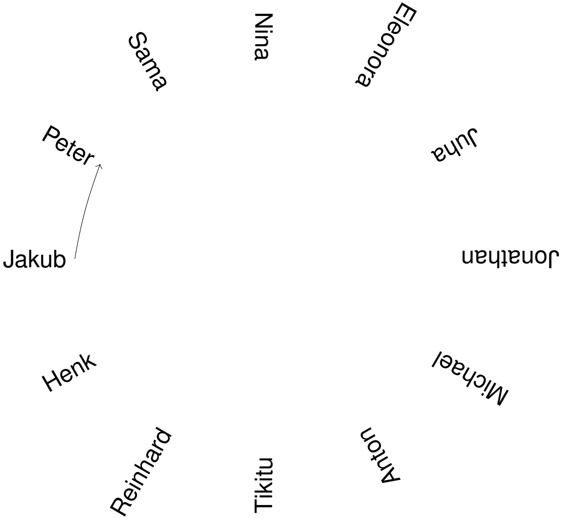
Quantifiers are useful

Everyone knows everyone here.



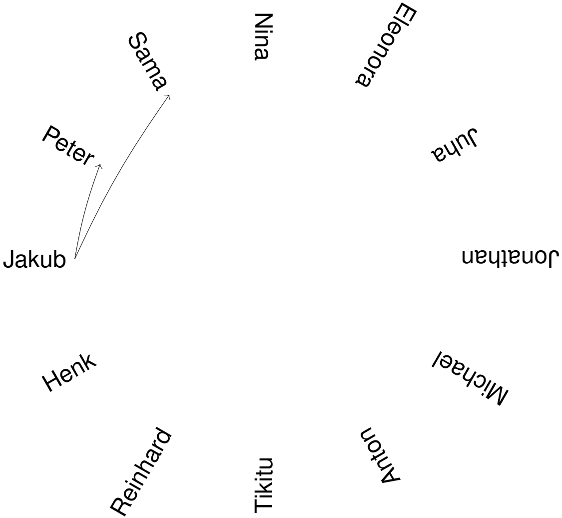
Quantifiers are useful

Everyone knows everyone here.



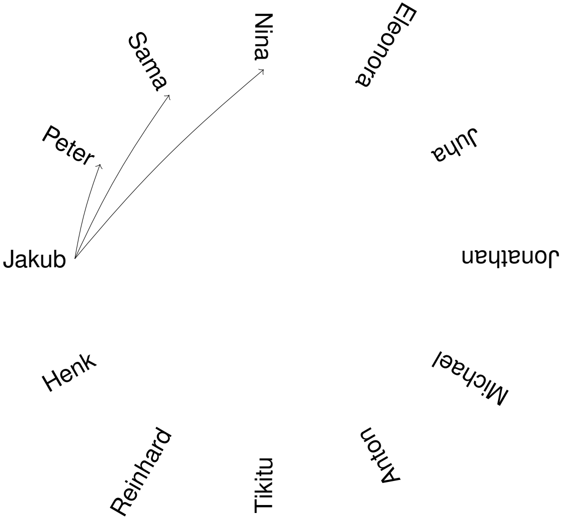
Quantifiers are useful

Everyone knows everyone here.



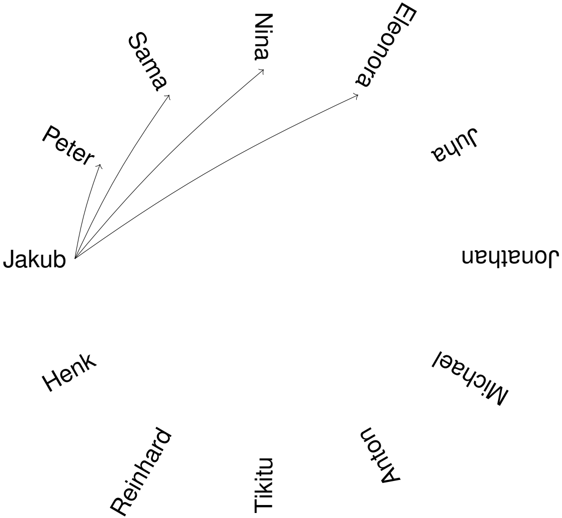
Quantifiers are useful

Everyone knows everyone here.



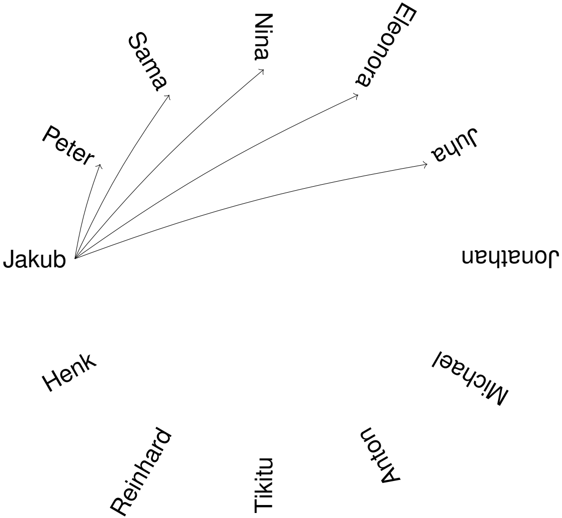
Quantifiers are useful

Everyone knows everyone here.



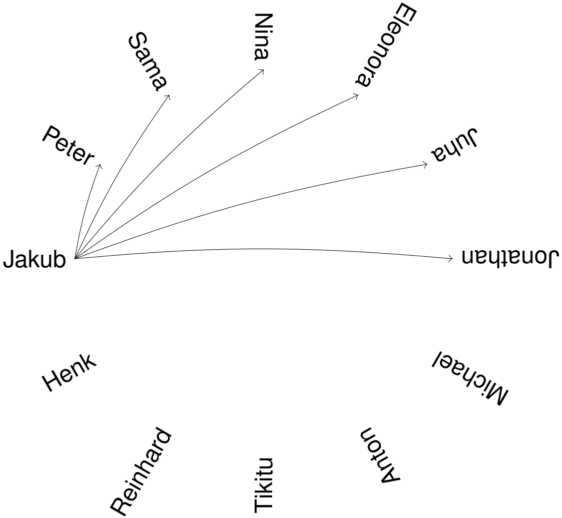
Quantifiers are useful

Everyone knows everyone here.



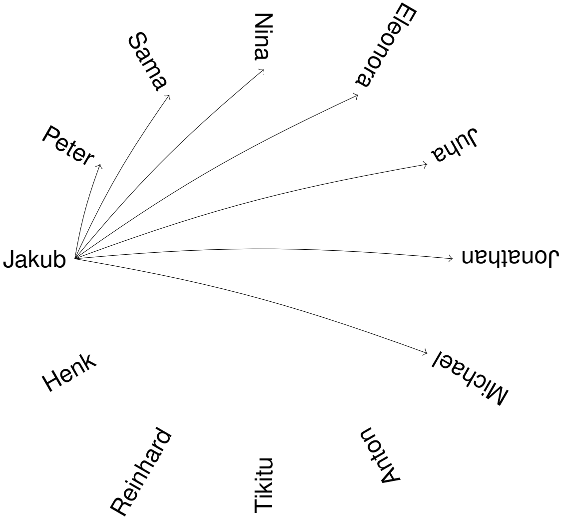
Quantifiers are useful

Everyone knows everyone here.



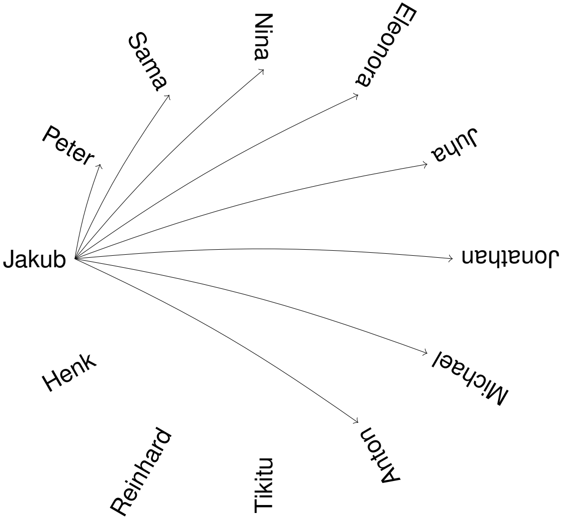
Quantifiers are useful

Everyone knows everyone here.



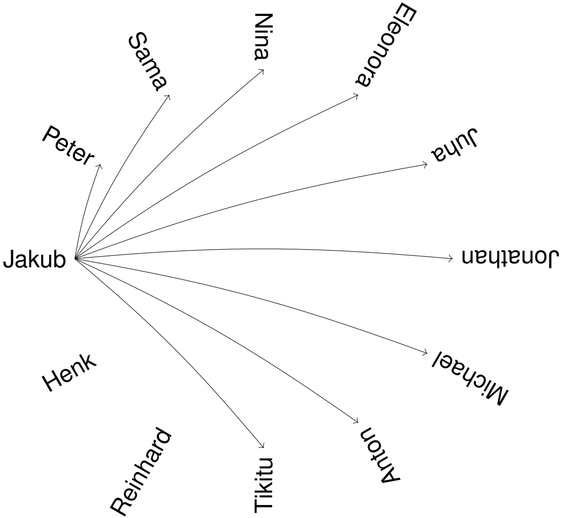
Quantifiers are useful

Everyone knows everyone here.



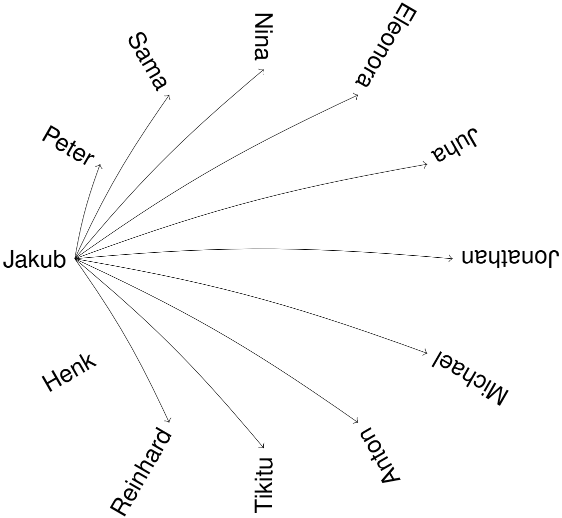
Quantifiers are useful

Everyone knows everyone here.



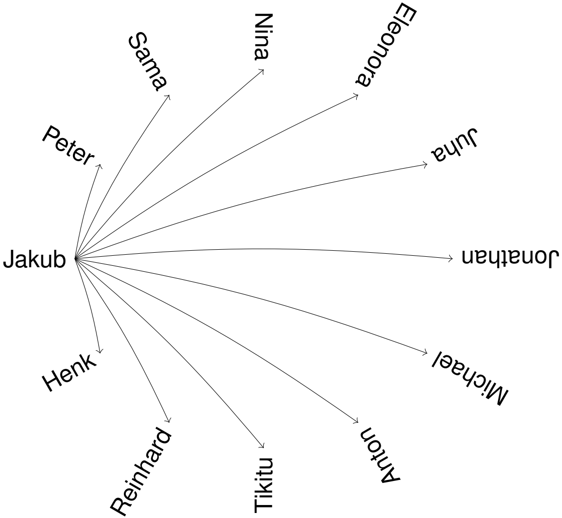
Quantifiers are useful

Everyone knows everyone here.



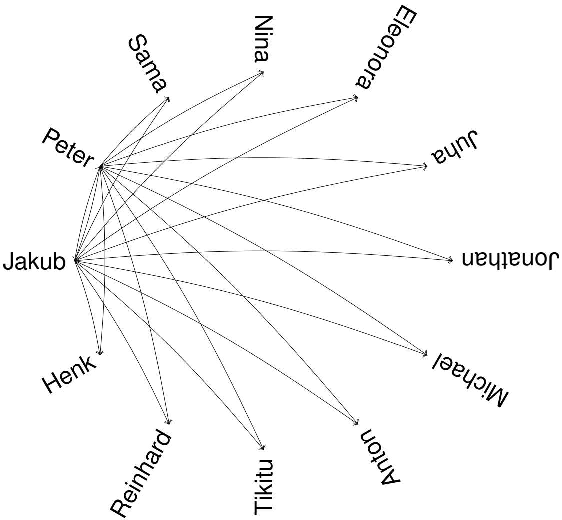
Quantifiers are useful

Everyone knows everyone here.



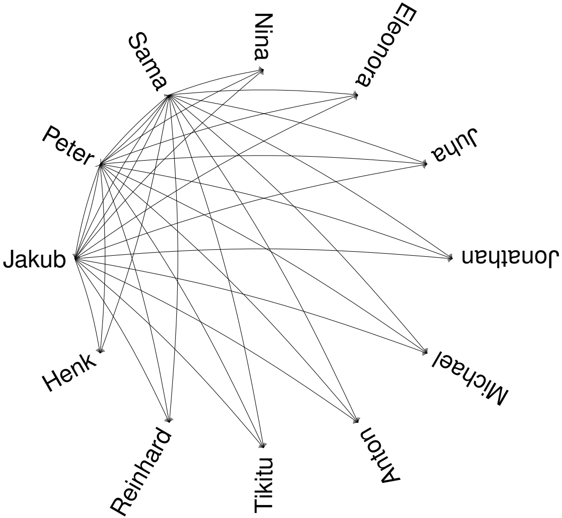
Quantifiers are useful

Everyone knows everyone here.



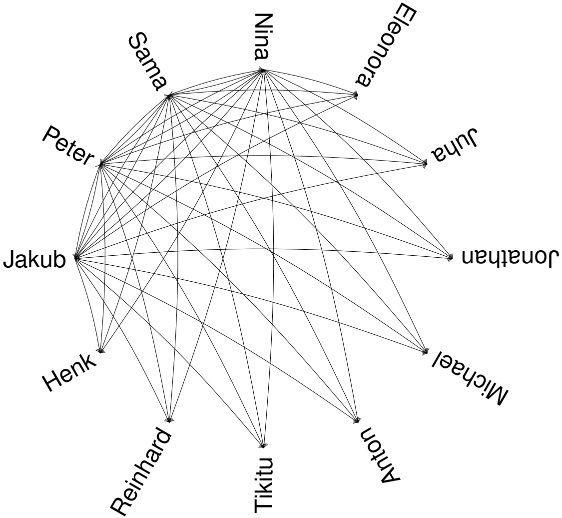
Quantifiers are useful

Everyone knows everyone here.



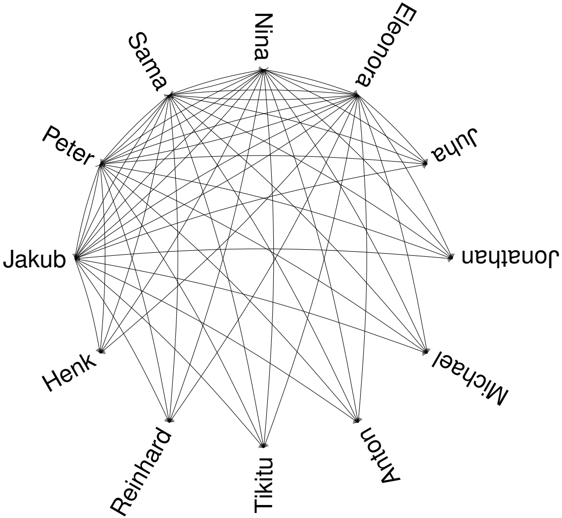
Quantifiers are useful

Everyone knows everyone here.



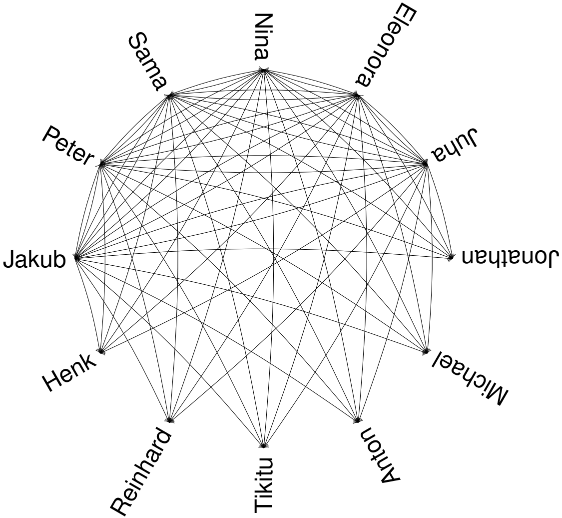
Quantifiers are useful

Everyone knows everyone here.



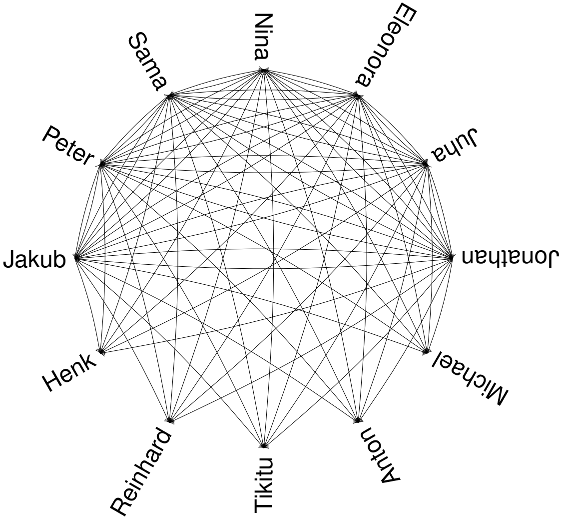
Quantifiers are useful

Everyone knows everyone here.



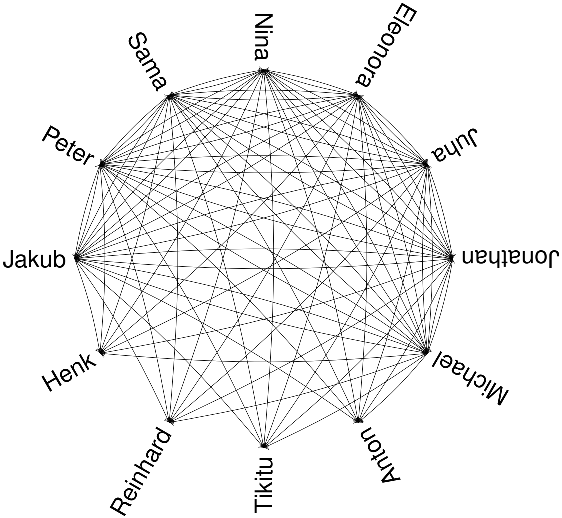
Quantifiers are useful

Everyone knows everyone here.



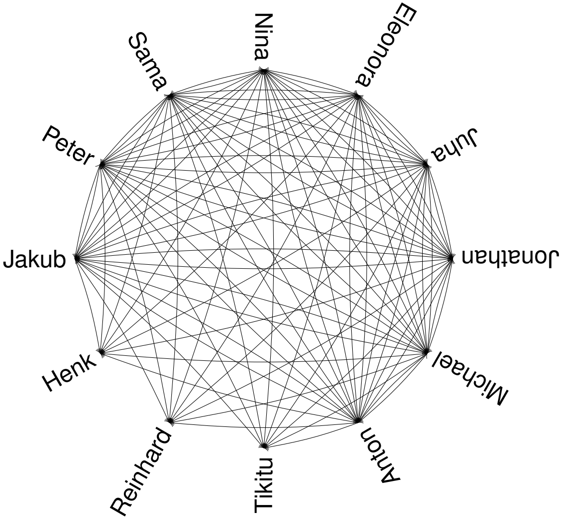
Quantifiers are useful

Everyone knows everyone here.



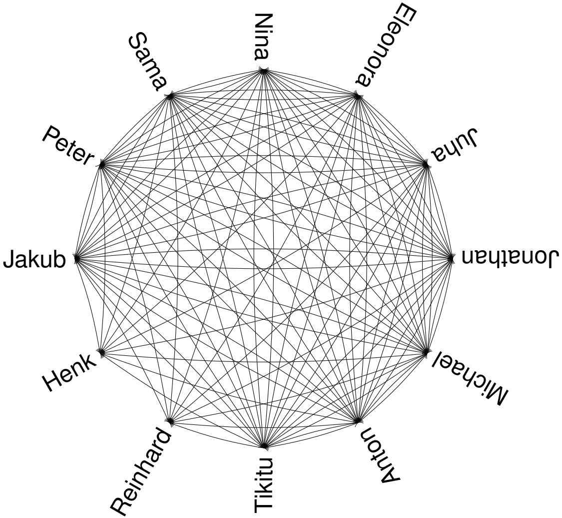
Quantifiers are useful

Everyone knows everyone here.



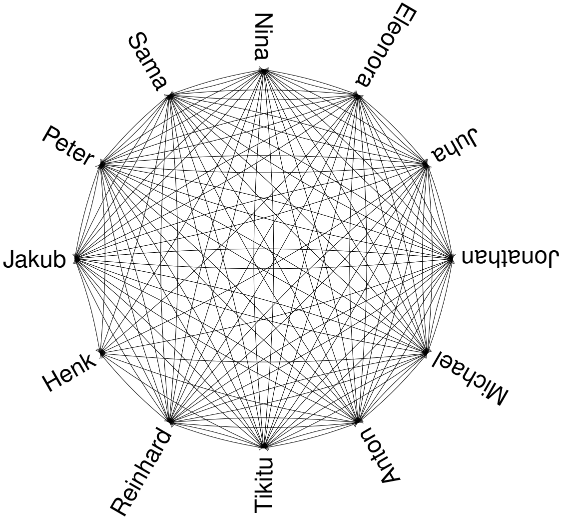
Quantifiers are useful

Everyone knows everyone here.



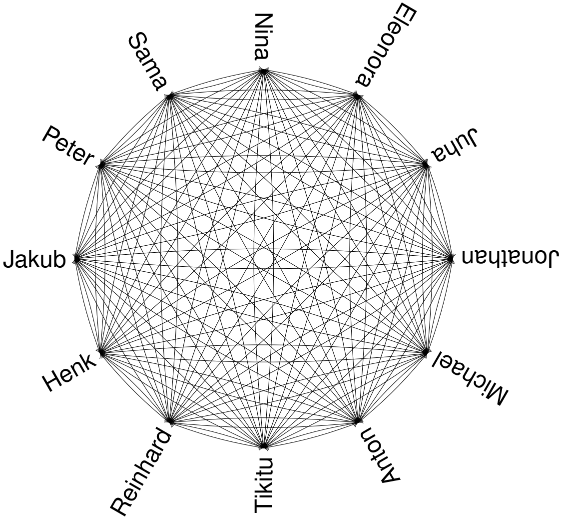
Quantifiers are useful

Everyone knows everyone here.



Quantifiers are useful

Everyone knows everyone here.



Literature

- Westerståhl, Generalized Quantifiers, SEP.
- Peters & Westerståhl, Quantifiers in Language and Logic, OUP, 2008.
- Handbook of Logic and Language, 2nd edition, Van Benthem & Ter Meulen (Eds.), Elsevier 2011.
- `http://jakubszymanik.com/EGG2013/`
- `http://www.jakubszymanik.com/`

Outline

Generalized Quantifiers

Semantic universale

Monotonicity patterns

GQs in logic

Languages and automata

Computing quantifiers

Computational Complexity

Complex GQs
- Polyadic quantifiers
  - Branching Quantifiers
  - Strong Reciprocity
- Collective quantifiers

1. All poets have low self-esteem.
2. Some dean danced nude on the table.
3. At least 3 grad students prepared presentations.
4. An even number of the students saw a ghost.
5. Most of the students think they are smart.
6. Less than half of the students received good marks.
7. Many of the soldiers have not eaten for several days.
8. A few of the conservatives complained about taxes.

Determiners

Definition

Expressions that appear to be descriptions of quantity.

Example

All, not quite all, nearly all, an awful lot, a lot, a comfortable majority, most, many, more than *n*, less than *n*, quite a few, quite a lot, several, not a lot, not many, only a few, few, a few, hardly any, one, two, three.

Quantifiers are second-order relations

Observation

If we fix a model $\mathbb{M} = (M, A^M, B^M)$, then we can treat a generalized quantifier as a relation between relations over the universe.

Example

$$\text{every}[A, B] = 1 \text{ iff } A^M \subseteq B^M$$

Quantifiers are second-order relations

Observation

If we fix a model $\mathbb{M} = (M, A^M, B^M)$, then we can treat a generalized quantifier as a relation between relations over the universe.

Example

$$\mathsf{every}[A, B] = 1 \text{ iff } A^M \subseteq B^M$$

$$\mathsf{even}[A, B] = 1 \text{ iff } \mathsf{card}(A^M \cap B^M) \text{ is even}$$

Quantifiers are second-order relations

Observation

If we fix a model $\mathbb{M} = (M, A^M, B^M)$, then we can treat a generalized quantifier as a relation between relations over the universe.

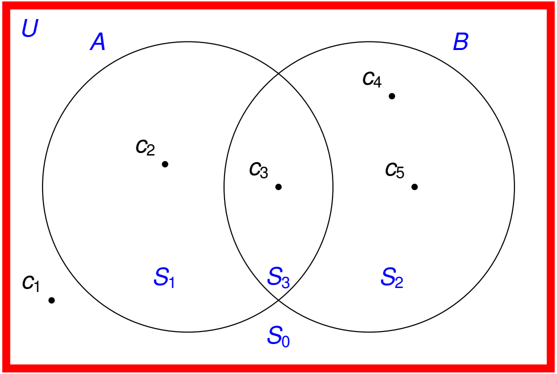
Example

$$\mathsf{every}[A, B] = 1 \text{ iff } A^M \subseteq B^M$$

$$\mathsf{even}[A, B] = 1 \text{ iff } \mathsf{card}(A^M \cap B^M) \text{ is even}$$

$$\mathsf{most}[A, B] = 1 \text{ iff } \mathsf{card}(A^M \cap B^M) > \mathsf{card}(A^M - B^M)$$

Illustration



Generalized Quantifiers

Definition

A quantifier Q is a way of associating with each set M a function from pairs of subsets of M into $\{0, 1\}$ (False, True).

Example

$$\text{every}_M[A, B] = 1 \text{ iff } A \subseteq B$$

Generalized Quantifiers

Definition

A quantifier Q is a way of associating with each set M a function from pairs of subsets of M into $\{0, 1\}$ (False, True).

Example

$$\mathrm{every}_M[A, B] = 1 \text{ iff } A \subseteq B$$

$$\mathrm{even}_M[A, B] = 1 \text{ iff } \mathrm{card}(A \cap B) \text{ is even}$$

Generalized Quantifiers

Definition

A quantifier Q is a way of associating with each set M a function from pairs of subsets of M into $\{0, 1\}$ (False, True).

Example

$$\text{every}_M[A, B] = 1 \text{ iff } A \subseteq B$$

$$\text{even}_M[A, B] = 1 \text{ iff } \text{card}(A \cap B) \text{ is even}$$

$$\text{most}_M[A, B] = 1 \text{ iff } \text{card}(A \cap B) > \text{card}(A - B)$$

Space of GQs

- If $\mathrm{card}(M) = n$, then there are $2^{2^{2n}}$ GQs.
- For $n = 2$ it gives 65,536 possibilities.

Space of GQs

- If $\mathrm{card}(M) = n$, then there are $2^{2^{2n}}$ GQs.
- For $n = 2$ it gives 65,536 possibilities.

Question

Which of those correspond to simple determiners?

Outline

Generalized Quantifiers

Semantic universale

Monotonicity patterns

GQs in logic

Languages and automata

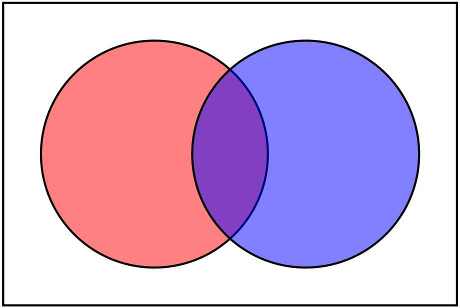
Computing quantifiers

Computational Complexity

Complex GQs
- Polyadic quantifiers
  - Branching Quantifiers
  - Strong Reciprocity
- Collective quantifiers

Isomorphism closure

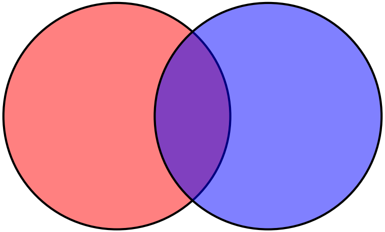
(ISOM) If $(M, A, B) \cong (M', A', B')$, then $\mathrm{Q}_{\mathbf{M}}(A, B) \Leftrightarrow \mathrm{Q}_{\mathbf{M}'}(A', B')$



Topic neutrality

Extensionality

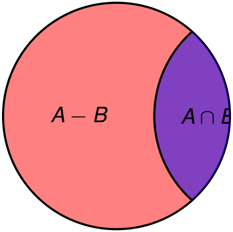
(EXT) If $M \subseteq M'$, then $Q_{\mathbf{M}}(A, B) \Leftrightarrow Q_{\mathbf{M'}}(A, B)$



Universe independence

Conservativity

(CONS) $\mathsf{Q}_{\mathbf{M}}(A, B) \Leftrightarrow \mathsf{Q}_{\mathbf{M}}(A, A \cap B)$



Research questions

- Do all NL determiners satisfy ISOM, EXT and CONS?
- Only?
- If yes, why?
  1. Learnability?
  2. Evolution?

Hunter & Lidz, Conservativity and learnability of determiners, Journal of Semantics, 2010

Kids...

1. There are blue non-circles . . . Are all the circle blue?
2. There are elephants not being ridden by a girl . . .
   Is every girl riding an elephant?

Gleeb and gleeb'

$$\text{gleeb}_M[A, B] = 1 \text{ iff } A \not\subseteq B$$

$$\text{gleeb}_M[A, B] = 1 \text{ iff } B \not\subseteq A$$

(a)

(b)

Experiment

- Picky puppet task
- The puppet told me that he likes this card because gleeb girls are on the beach.
- The puppet told me that he doesn't like this card because it's not true that gleeb girls are on the beach.

Condition	Conservative	Nonconservative
Cards correctly sorted (out of 5)	mean 4.1 (above chance, $p < 0.0001$)	mean 3.1 (not above chance, $p > 0.2488$)
Subjects with "perfect" accuracy	50%	10%

Number triangle representation

					(0,0)					
				(1,0)		(0,1)				
			(2,0)		(1,1)		(0,2)			
		(3,0)		(2,1)		(1,2)		(0,3)		
	(4,0)		(3,1)		(2,2)		(1,3)		(0,4)	
...										...

Number triangle representation

-

- +

- + +

- + + +

- + + + +

... ...

General definition

Definition

A monadic generalized quantifier of type (1,1) is a class Q of structures of the form $M = (U, A_1, A_2)$, where $A_1, A_2 \subseteq U$. Additionally, Q is closed under isomorphism.

Examples

$$\text{every} = \{(M, A, B) \mid A, B \subseteq M \text{ and } A \subseteq B\}.$$

Examples

$$\text{every} = \{(M, A, B) \mid A, B \subseteq M \text{ and } A \subseteq B\}.$$

$$\text{some} = \{(M, A, B) \mid A, B \subseteq M \text{ and } A \cap B \neq \emptyset\}.$$

Examples

$$\text{every} = \{(M, A, B) \mid A, B \subseteq M \text{ and } A \subseteq B\}.$$

$$\text{some} = \{(M, A, B) \mid A, B \subseteq M \text{ and } A \cap B \neq \emptyset\}.$$

$$\text{more than k} = \{(M, A, B) \mid A, B \subseteq M \text{ and } \operatorname{card}(A \cap B) > k\}.$$

Examples

$$\text{every} = \{(M, A, B) \mid A, B \subseteq M \text{ and } A \subseteq B\}.$$

$$\text{some} = \{(M, A, B) \mid A, B \subseteq M \text{ and } A \cap B \neq \emptyset\}.$$

$$\text{more than k} = \{(M, A, B) \mid A, B \subseteq M \text{ and } \text{card}(A \cap B) > k\}.$$

$$\text{even} = \{(M, A, B) \mid A, B \subseteq M \text{ and } \text{card}(A \cap B) \text{ is even}\}.$$

Examples

$$\text{every} = \{(M, A, B) \mid A, B \subseteq M \text{ and } A \subseteq B\}.$$

$$\text{some} = \{(M, A, B) \mid A, B \subseteq M \text{ and } A \cap B \neq \emptyset\}.$$

$$\text{more than k} = \{(M, A, B) \mid A, B \subseteq M \text{ and } \operatorname{card}(A \cap B) > k\}.$$

$$\text{even} = \{(M, A, B) \mid A, B \subseteq M \text{ and } \operatorname{card}(A \cap B) \text{ is even}\}.$$

$$\text{most} = \{(M, A, B) \mid A, B \subseteq M \text{ and } \operatorname{card}(A \cap B) > \operatorname{card}(A - B)\}$$

Outline

Generalized Quantifiers

Semantic universale

Monotonicity patterns

GQs in logic

Languages and automata

Computing quantifiers

Computational Complexity

Complex GQs
- Polyadic quantifiers
  - Branching Quantifiers
  - Strong Reciprocity
- Collective quantifiers

Monotonicity

↑MON $Q_M[A, B]$ and $A \subseteq A' \subseteq M$ then $Q_M[A', B]$.
↓MON $Q_M[A, B]$ and $A' \subseteq A \subseteq M$ then $Q_M[A', B]$.
MON↑ $Q_M[A, B]$ and $B \subseteq B' \subseteq M$ then $Q_M[A, B']$.
MON↓ $Q_M[A, B]$ and $B' \subseteq B \subseteq M$ then $Q_M[A, B']$.

Inference test

1. Some boy is dirty.
2. Some child is dirty.

1. All child is dirty.
2. All boy is dirty.

1. All boy is muddy.
2. All boy is dirty.

1. No boy is dirty.
2. No boy is muddy.

1. Exactly five children are dirty.
2. Exactly five boys are dirty.

Boolean combinations

1. At least 5 or at most 10 departments can win EU grants. (disjunction)
2. Between 100 and 200 students started in the marathon. (conjunction)
3. Not all students passed. (outer negation)
4. All students did not pass. (inner negation)

Definition

$$(\mathrm{Q} \wedge \mathrm{Q}')_M[A, B] \iff \mathrm{Q}_M[A, B] \text{ and } \mathrm{Q}'_M[A, B] \text{ (conjunction)}$$
$$(\mathrm{Q} \vee \mathrm{Q}')_M[A, B] \iff \mathrm{Q}_M[A, B] \text{ or } \mathrm{Q}'_M[A, B] \text{ (disjunction).}$$
$$(\neg \mathrm{Q})_M[A, B] \iff \text{not } \mathrm{Q}_M[A, B] \text{ (complement)}$$
$$(\mathrm{Q}\neg)_M[A, B] \iff Q_M[A, M - B] \text{ (post-complement)}$$

Monotonicity interacts with negation

Theorem

Q is MON↑

1. *iff ¬Q is MON↓.*
2. *iff Q¬ is MON↓.*

Q is ↑MON

1. *iff ¬Q is ↓MON.*
2. *iff Q¬ is ↑MON.*

Similarly for the downward monotone case.

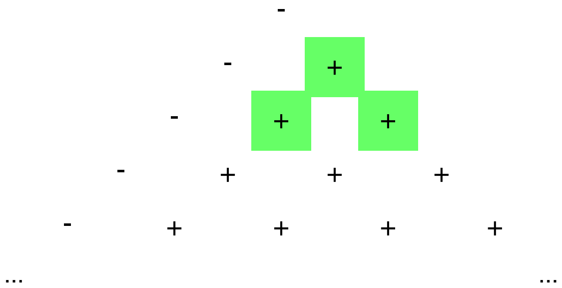
Square of opposition

- some, $\neg$ some $=$ no, some $\neg$ $=$ not all, $\neg$ some $\neg$ $=$ all .
- some is $\uparrow$MON$\uparrow$.
- Therefore, no is $\downarrow$MON$\downarrow$, not all is $\uparrow$MON$\downarrow$, and all is $\downarrow$MON$\uparrow$.

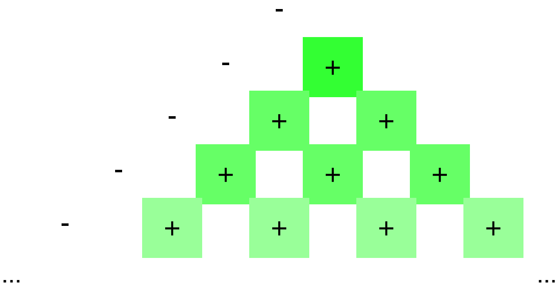
↑MON

$$\begin{array}{ccccccccccc}
 & & & & & - & & & & & \\
 & & & & - & & \boxed{+} & & & & \\
 & & & - & & + & & + & & & \\
 & & - & & + & & + & & + & & \\
 & - & & + & & + & & + & & + & \\
\dots & & & & & & & & & & \dots
\end{array}$$

↑MON



↑MON



It even helps to find an efficient solution for:

It even helps to find an efficient solution for:

Gierasimczuk & Szymanik, Invariance properties of quantifiers and multiagent information exchange, Proc. of 12th Meeting on Mathematics of Language

Outline

Generalized Quantifiers

Semantic universale

Monotonicity patterns

GQs in logic

Languages and automata

Computing quantifiers

Computational Complexity

Complex GQs
- Polyadic quantifiers
  - Branching Quantifiers
  - Strong Reciprocity
- Collective quantifiers

Definability

Definition

Let Q be a generalized quantifier and $\mathcal{L}$ a logic. We say that the quantifier Q is *definable* in $\mathcal{L}$ if there is a sentence $\varphi \in \mathcal{L}$ such that for any $\mathbb{M}$:

$$\mathbb{M} \models \varphi \text{ iff } \mathsf{Q}_M[A, B].$$

Elementary GQs

Some GQs, like $\exists^{\leq 3}$, $\exists^{=3}$, and $\exists^{\geq 3}$, are expressible in FO.

Example

$$\text{some } x\, [A(x), B(x)] \iff \exists x[A(x) \wedge B(x)].$$

Non-elementary GQs

Theorem

The quantifiers 'there exists (in)finitely many', most *and* even *are not first-order definable.*

Non-elementary GQs

Theorem

The quantifiers 'there exists (in)finitely many', most *and* even *are not first-order definable.*

We can use higher-order logics:

Non-elementary GQs

Theorem

The quantifiers 'there exists (in)finitely many', most *and* even *are not first-order definable.*

We can use higher-order logics:

Example

In $\mathbb{M} = (M, A^M, B^M)$ the sentence

$$\text{most } x\ [A(x), B(x)]$$

is true if and only if the following condition holds:

$$\exists f : (A^M - B^M) \longrightarrow (A^M \cap B^M) \text{ such that f is injective but not surjective.}$$

Theorem (Westerståhl 1998)

In finite models, persistent quantifiers satisfying EXT, ISOM and CONS are FO-definable.

Outline

Generalized Quantifiers

Semantic universale

Monotonicity patterns

GQs in logic

Languages and automata

Computing quantifiers

Computational Complexity

Complex GQs
- Polyadic quantifiers
  - Branching Quantifiers
  - Strong Reciprocity
- Collective quantifiers

Languages - basic definitions

- *Alphabet* is any non-empty finite set of symbols, e.g., $A = \{a, b\}$ and $B = \{0, 1\}$.

Languages - basic definitions

- *Alphabet* is any non-empty finite set of symbols, e.g., $A = \{a, b\}$ and $B = \{0, 1\}$.
- A *word (string)* is a finite sequence of symbols from a given alphabet, e.g., “1110001110”.

Languages - basic definitions

- *Alphabet* is any non-empty finite set of symbols, e.g., $A = \{a, b\}$ and $B = \{0, 1\}$.
- A *word (string)* is a finite sequence of symbols from a given alphabet, e.g., “1110001110”.
- The *empty word*, ε, is a sequence without symbols.

Languages - basic definitions

- *Alphabet* is any non-empty finite set of symbols, e.g., $A = \{a, b\}$ and $B = \{0, 1\}$.
- A *word (string)* is a finite sequence of symbols from a given alphabet, e.g., "1110001110".
- The *empty word*, ε, is a sequence without symbols.
- The *length of a word* is the number of symbols in it.

Languages - basic definitions

- *Alphabet* is any non-empty finite set of symbols, e.g., $A = \{a, b\}$ and $B = \{0, 1\}$.
- A *word (string)* is a finite sequence of symbols from a given alphabet, e.g., "1110001110".
- The *empty word*, ε, is a sequence without symbols.
- The *length of a word* is the number of symbols in it.
- *The set of all words over alphabet* Γ is denoted by Γ^*, e.g., $\{0, 1\}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$.

Languages - basic definitions

- *Alphabet* is any non-empty finite set of symbols, e.g., $A = \{a, b\}$ and $B = \{0, 1\}$.
- A *word (string)* is a finite sequence of symbols from a given alphabet, e.g., “1110001110”.
- The *empty word*, ε, is a sequence without symbols.
- The *length of a word* is the number of symbols in it.
- *The set of all words over alphabet* Γ is denoted by Γ^*, e.g., $\{0, 1\}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$.
- Any set of words, a subset of Γ^*, will be called a *language*.

Finite automata

Definition

A *non-deterministic finite automaton* (FA) is a tuple (A, Q, q_s, F, δ), where:

- A is an input alphabet;
- Q is a finite set of states;
- $q_s \in Q$ is an initial state;
- $F \subseteq Q$ is a set of accepting states;
- $\delta : Q \times A \longrightarrow \mathcal{P}(Q)$ is a transition function.

Regular languages

Definition

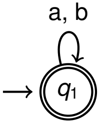
The language accepted (recognized) by some FA H, L(H), is the set of all words over the alphabet A which are accepted by H.

Definition

We say that a *language $L \subseteq A^*$ is regular* if and only if there exists some FA H such that $L = L(H)$.

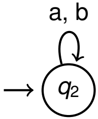
Example 1

Let $A = \{a, b\}$ and consider the language $L_1 = A^*$.



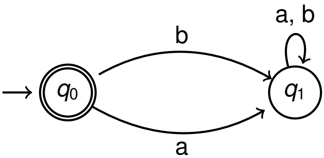
Example 2

Let $L_2 = \emptyset$



Example 3

$L_3 = \{\varepsilon\}$



Not every language is regular

$$L_{ab} = \{a^n b^n : n \geq 1\}$$

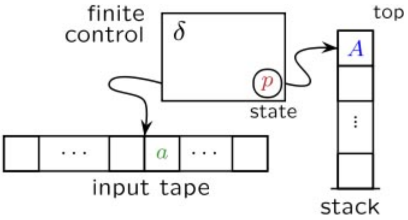
Push down automata

Definition

A non-deterministic push-down automaton (PDA) is a tuple $(A, \Gamma, \#, Q, q_s, F, \delta)$, where:

- A is an input alphabet;
- Γ is a stack alphabet;
- $\# \notin \Gamma$ is a stack initial symbol, empty stack consists only from it;
- Q is a finite set of states;
- $q_s \in Q$ is an initial state;
- $F \subseteq Q$ is a set of accepting states;
- $\delta : Q \times (A \cup \{\varepsilon\}) \times \Gamma \longrightarrow \mathcal{P}(Q \times \Gamma^*)$ is a transition function.

PDA



push/pop-off a symbol from the top of the stack

Context-free languages

Definition
We say that a language $L \subseteq A^*$ is context-free if and only if there is a PDA H such that $L = L(H)$.

Regular $\subset$ context-free

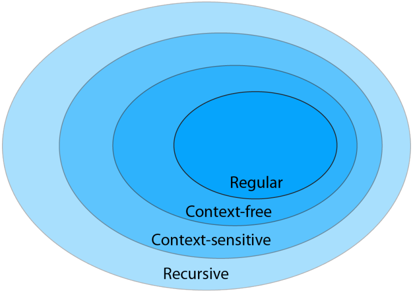
There is a PDA for $L_{ab} = \{a^n b^n : n \geq 1\}$.

Beyond context-free languages

$$L_{abc} = \{a^k b^k c^k : k \geq 1\}$$

We will investigate stronger languages in the last lecture.

Chomsky's Hierarchy



Outline

Generalized Quantifiers

Semantic universale

Monotonicity patterns

GQs in logic

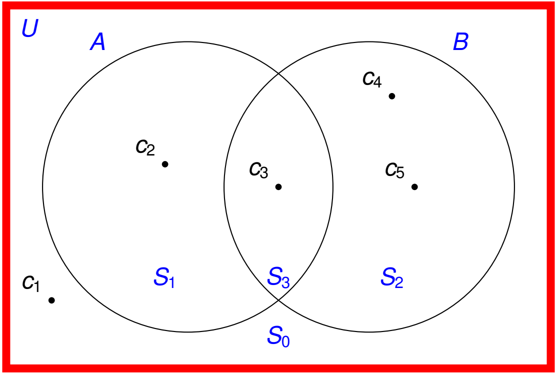
Languages and automata

Computing quantifiers

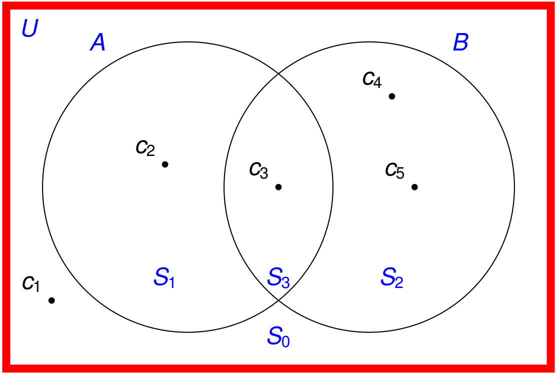
Computational Complexity

Complex GQs
- Polyadic quantifiers
  - Branching Quantifiers
  - Strong Reciprocity
- Collective quantifiers

General definition



How do we encode models?



This model is uniquely described by $\alpha_M = a_{\bar{A}\bar{B}} a_{A\bar{B}} a_{AB} a_{\bar{A}B} a_{\bar{A}B}$

Step by step

- Restriction to finite models of the form $M = (U, A, B)$.

Step by step

- Restriction to finite models of the form $M = (U, A, B)$.
- List of all elements of the model: $c_1, \ldots, c_5$.

Step by step

- Restriction to finite models of the form $M = (U, A, B)$.
- List of all elements of the model: $c_1, \ldots, c_5$.
- Labeling every element with one of the letters:
  $a_{\bar{A}\bar{B}}$, $a_{A\bar{B}}$, $a_{\bar{A}B}$, a_{AB}, according to constituents it belongs to.

Step by step

- Restriction to finite models of the form $M = (U, A, B)$.
- List of all elements of the model: $c_1, \ldots, c_5$.
- Labeling every element with one of the letters:
  $a_{\bar{A}\bar{B}}$, $a_{A\bar{B}}$, $a_{\bar{A}B}$, a_{AB}, according to constituents it belongs to.
- Result: the word $\alpha_M = a_{\bar{A}\bar{B}} a_{A\bar{B}} a_{AB} a_{\bar{A}B} a_{\bar{A}B}$.

Step by step

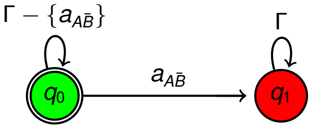
- Restriction to finite models of the form $M = (U, A, B)$.
- List of all elements of the model: $c_1, \ldots, c_5$.
- Labeling every element with one of the letters: $a_{\bar{A}\bar{B}}$, $a_{A\bar{B}}$, $a_{\bar{A}B}$, a_{AB}, according to constituents it belongs to.
- Result: the word $\alpha_M = a_{\bar{A}\bar{B}} a_{A\bar{B}} a_{AB} a_{\bar{A}B} a_{\bar{A}B}$.
- α_M describes the model in which: $c_1 \in \bar{A}\bar{B}$, $c_2 \in A\bar{B} c_3 \in AB$, $c_4 \in \bar{A}B$, $c_5 \in \bar{A}B$.

Step by step

- Restriction to finite models of the form $M = (U, A, B)$.
- List of all elements of the model: $c_1, \ldots, c_5$.
- Labeling every element with one of the letters:
  $a_{\bar{A}\bar{B}}$, $a_{A\bar{B}}$, $a_{\bar{A}B}$, a_{AB}, according to constituents it belongs to.
- Result: the word $\alpha_M = a_{\bar{A}\bar{B}}a_{A\bar{B}}a_{AB}a_{\bar{A}B}a_{\bar{A}B}$.
- α_M describes the model in which:
  $c_1 \in \bar{A}\bar{B}$, $c_2 \in A\bar{B}c_3 \in AB$, $c_4 \in \bar{A}B$, $c_5 \in \bar{A}B$.
- The class Q is represented by the set of words describing all elements of the class.

Aristotelian quantifiers

“all”, “some”, “no”, and “not all”

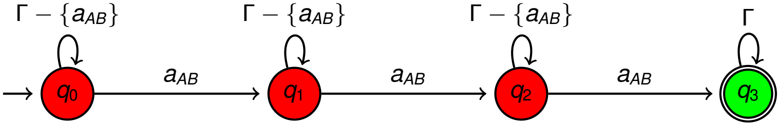


Finite automaton recognizing L_{All}

$$L_{\text{All}} = \{\alpha \in \Gamma^* : \#a_{A\bar{B}}(\alpha) = 0\}$$

Cardinal quantifiers

E.g. "more than 2", "less than 7", and "between 8 and 11"

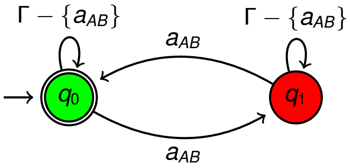


Finite automaton recognizing $L_{\text{More than two}}$

$$L_{\text{More than two}} = \{\alpha \in \Gamma^* : \#a_{AB}(\alpha) > 2\}$$

Parity quantifiers

E.g. “an even number”, “an odd number”

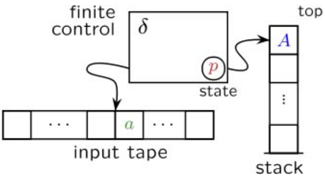


Finite automaton recognizing L_{Even}

$$L_{\text{Even}} = \{\alpha \in \Gamma^* : \#a_{AB}(\alpha) \text{ is even}\}$$

Proportional quantifiers

- E.g. “most”, “less than half”.
- Most *As are B* iff $\text{card}(A \cap B) > \text{card}(A - B)$.
- $L_{\text{Most}} = \{\alpha \in \Gamma^* : \#a_{AB}(\alpha) > \#a_{A\bar{B}}(\alpha)\}$.
- There is no finite automaton recognizing this language.
- We need internal memory.
- A push-down automata will do.



Summing up

Definability	Examples	Recognized by
FO	“all” “at least 3”	acyclic FA
FO(D_n)	“an even number”	FA
PrA	“most”, “less than half”	PDA

Quantifiers, definability, and complexity of automata

Van Benthem, Essays in logical semantics, 1986.

Mostowski, Computational semantics for monadic quantifiers, 1998.

Does it say anything about processing?

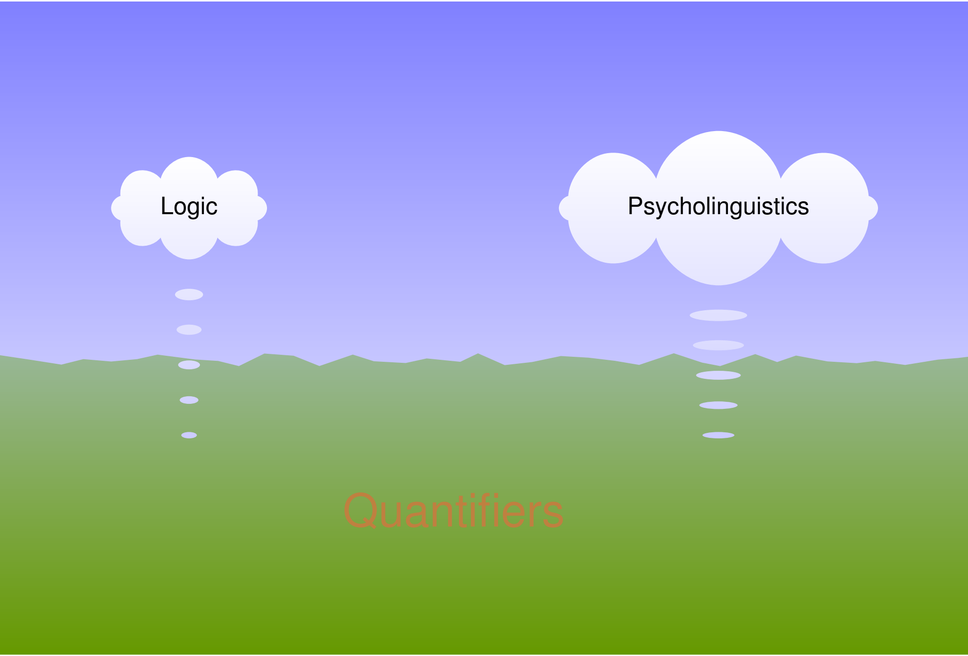
Question

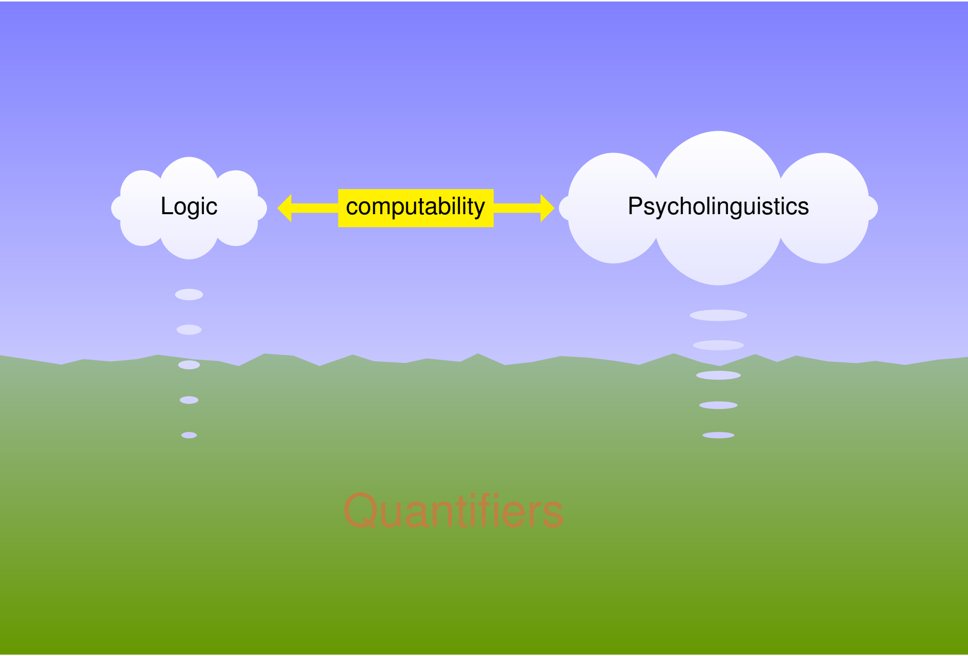
Do minimal automata predict differences in verification?

Quantifiers

Logic

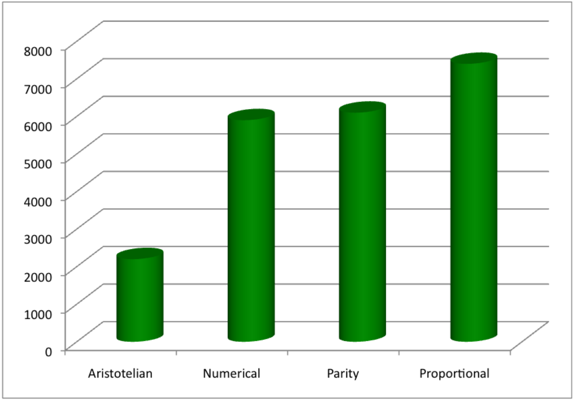
Quantifiers


Logic
Psycholinguistics
Quantifiers


Logic
computability
Psycholinguistics
Quantifiers

A simple study

More than half of the cars are yellow.



Szymaniki & Zajenkowski, Comprehension of simple quantifiers. Empirical evaluation of a computational model, Cognitive Science, 2010

Neurobehavioral studies

Differences in brain activity.

- All quantifiers are associated with numerosity:
  recruit right inferior parietal cortex.
- Only higher-order activate working-memory capacity:
  recruit right dorsolateral prefrontal cortex.

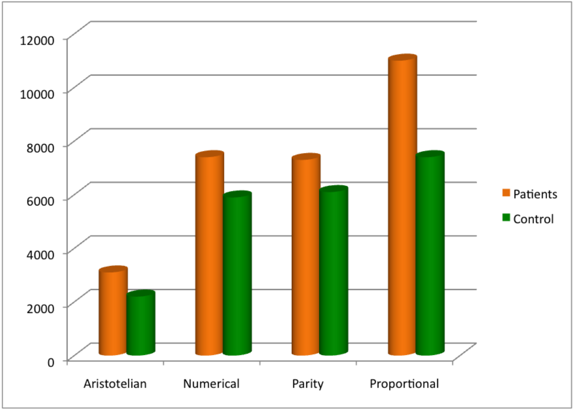
McMillan et al., Neural basis for generalized quantifiers comprehension, Neuropsychologia, 2005

Szymanik, A Note on some neuroimaging study of natural language quantifiers comprehension, Neuropsychologia, 2007

Experiment with schizophrenic patients

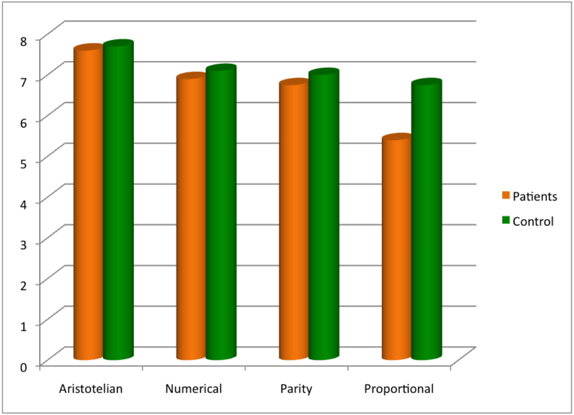
- Compare performance of:
  - Healthy subjects.
  - Patients with schizophrenia.
    - Known WM deficits.

RT data



12000
10000
8000
6000
4000
2000
0
Aristotelian
Numerical
Parity
Proportional
Patients
Control

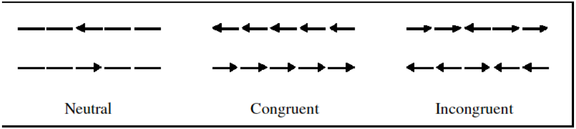
Accuracy data



Zajenkowski et al., A computational approach to quantifiers as an explanation for some language impairments in schizophrenia, Journal of Communication Disorders, 2011.

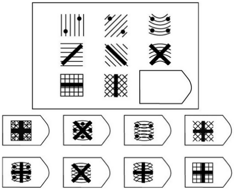
Teasing apart WM from executive control

- Executive control within Attention Networks Test (ANT)
  - resolution of conflict between expectation, stimulus, and response
  - = incongruent flanking – congruent flanking



+ Intelligence

- Quantifier verification + Sternberg's STM +ANT
- Raven's Advanced Progressive Matrices Test (APM)
    1. test of fluid intelligence
    2. find a missing one



Quantifiers, WM, and intelligence

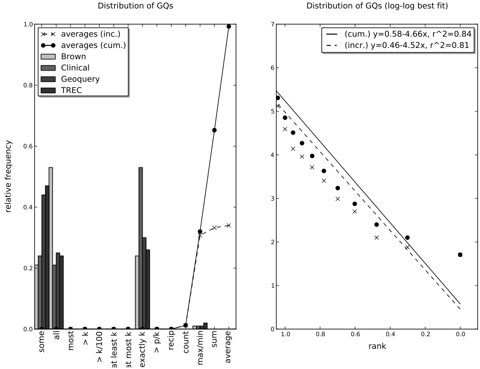
- All quantifier correlated with STM.
- Only proportional quantifiers correlated with ANT.
- APM correlated best with proportional quantifiers.
- APM attenuated ANT.

Zajenkowski and Szymanik. Most intelligent people are accurate and some fast people are intelligent, Intelligence 2013

Research questions

1. Build computational cognitive model
   - visual aspects
   - working memory model
   - pragmatics
   - etc.

Distribution is skewed towards quantifiers of low complexity



Thorne & Szymanik. Generalized Quantifier Distribution and Semantic Complexity, 2013.

Outline

Generalized Quantifiers

Semantic universale

Monotonicity patterns

GQs in logic

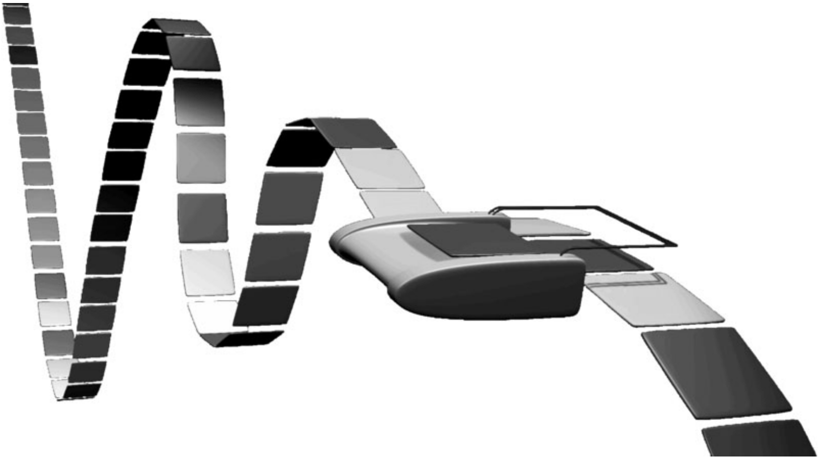
Languages and automata

Computing quantifiers

Computational Complexity

Complex GQs
- Polyadic quantifiers
  - Branching Quantifiers
  - Strong Reciprocity
- Collective quantifiers

Model of Computation



Computational Complexity Theory

Question

What amount of resources TM needs to solve a task?

Computational Complexity Theory

Question

What amount of resources TM needs to solve a task?

Theorem (determinism vs. non-determinism)

If there is a non-deterministic Turing machine N recognizing a language L, then there exists a deterministic Turing machine M for language L.

Question

What amount of resources TM needs to solve a task?

Theorem (determinism vs. non-determinism)

If there is a non-deterministic Turing machine N recognizing a language L, then there exists a deterministic Turing machine M for language L.

Question

The simulation takes $O(c^{f(n)})$. Can we do it significantly faster?

Time Complexity

Let $f : \omega \longrightarrow \omega$.

Time Complexity

Let $f : \omega \longrightarrow \omega$.

Definition

TIME(f) is the class of languages (problems) which can be recognized by a deterministic Turing machine in time bounded by f with respect to the length of the input.

Time Complexity

Let $f : \omega \longrightarrow \omega$.

Definition

TIME(f) is the class of languages (problems) which can be recognized by a deterministic Turing machine in time bounded by f with respect to the length of the input.

Definition

NTIME(f), is the class of languages L for which there exists a non-deterministic Turing machine M such that for every $x \in L$ all branches in the computation tree of M on x are bounded by $f(n)$ and moreover M decides L.

Complexity Classes P and NP

Definition

- PTIME $= \bigcup_{k \in \omega} \mathrm{TIME}(n^k)$
- NPTIME $= \bigcup_{k \in \omega} \mathrm{NTIME}(n^k)$

Complexity Classes P and NP

Definition

- $\mathrm{PTIME} = \bigcup_{k \in \omega} \mathrm{TIME}(n^k)$
- $\mathrm{NPTIME} = \bigcup_{k \in \omega} \mathrm{NTIME}(n^k)$

Question (Millenium Problem)

P=NP?

Definition

We say that a function $f : A \longrightarrow A$ is a *polynomial time computable function* iff there exits a deterministic Turing machine computing $f(w)$ for every $w \in A$ in polynomial time.

(In)tractability

Definition

We say that a function $f : A \longrightarrow A$ is a *polynomial time computable function* iff there exits a deterministic Turing machine computing $f(w)$ for every $w \in A$ in polynomial time.

Definition

A problem L is polynomial reducible to a problem L' if there is a polynomial time computable function such that

$$w \in L \iff f(w) \in L'.$$

(In)tractability

Definition

We say that a function $f : A \longrightarrow A$ is a *polynomial time computable function* iff there exits a deterministic Turing machine computing $f(w)$ for every $w \in A$ in polynomial time.

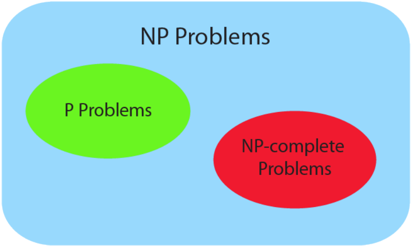
Definition

A problem L is polynomial reducible to a problem L' if there is a polynomial time computable function such that

$$w \in L \iff f(w) \in L'.$$

Definition

A language L is NP-complete if $L \in NP$ and every language in NP is reducible to L.


NP Problems
P Problems
NP-complete
Problems

Quantifiers in Finite Models

- Finite models can be encoded as strings.
- GQs as classes of such finite strings are languages.

Quantifiers in Finite Models

- Finite models can be encoded as strings.
- GQs as classes of such finite strings are languages.

Definition

By the *complexity of a quantifier* Q we mean the computational complexity of the corresponding class of finite models.

Question

$M \in \mathsf{Q}$? (equivalently $M \models Q$?)

Outline

Generalized Quantifiers

Semantic universale

Monotonicity patterns

GQs in logic

Languages and automata

Computing quantifiers

Computational Complexity

Complex GQs

- Polyadic quantifiers
  - Branching Quantifiers
  - Strong Reciprocity
- Collective quantifiers

Outline

Generalized Quantifiers

Semantic universale

Monotonicity patterns

GQs in logic

Languages and automata

Computing quantifiers

Computational Complexity

Complex GQs

- Polyadic quantifiers
  - Branching Quantifiers
  - Strong Reciprocity
- Collective quantifiers

Multi-quantifier sentences

1. Most villagers and most townsmen hate each other.
2. Three PMs referred to each other indirectly

$$\mathrm{Q}[A, B, R] \text{ or } \mathrm{Q}[A, R]$$

Lindström quantifiers

Definition

Let $t = (n_1, \ldots, n_k)$ be a k-tuple of positive integers. *A generalized quantifier* of type t is a class Q of models of a vocabulary $\tau_t = \{R_1, \ldots, R_k\}$, such that R_i is n_i-ary for $1 \leq i \leq k$, and Q is closed under isomorphisms.

Lindström quantifiers

Definition

Let $t = (n_1, \ldots, n_k)$ be a k-tuple of positive integers. *A generalized quantifier* of type t is a class Q of models of a vocabulary $\tau_t = \{R_1, \ldots, R_k\}$, such that R_i is n_i-ary for $1 \leq i \leq k$, and Q is closed under isomorphisms.

Definition

If in the above definition for all i: $n_i = 1$, then we say that a quantifier is *monadic*, otherwise we call it *polyadic*.

Lindström quantifiers

Definition

Let $t = (n_1, \ldots, n_k)$ be a k-tuple of positive integers. *A generalized quantifier* of type t is a class Q of models of a vocabulary $\tau_t = \{R_1, \ldots, R_k\}$, such that R_i is n_i-ary for $1 \leq i \leq k$, and Q is closed under isomorphisms.

Definition

If in the above definition for all i: $n_i = 1$, then we say that a quantifier is *monadic*, otherwise we call it *polyadic*.

$$
\begin{aligned}
\mathrm{W} &= \{(M, R) \mid R \subseteq M^2 \ \&\ R \text{ is a well-order}\}. \\
\mathrm{Ram} &= \{(M, A, R) \mid A \subseteq M, R \subseteq M^2 \ \&\ \forall a, b \in A\ R(a, b)\}.
\end{aligned}
$$

Coding

Definition

Let $\tau = \{R_1, \ldots, R_k\}$ be a relational vocabulary and $\mathbb{M}$ a τ-model of the following form: $\mathbb{M} = (U, R_1^M, \ldots, R_k^M)$, where $U = \{1, \ldots, n\}$ is the universe of model $\mathbb{M}$ and $R_i^M \subseteq U^{n_i}$ is an n_i-ary relation over U, for $1 \leq i \leq k$. We define a *binary encoding for τ-models*. The code for $\mathbb{M}$ is a word over $\{0, 1, \#\}$ of length $O((\mathrm{card}(U))^c)$, where c is the maximal arity of the predicates in τ (or $c = 1$ if there are no predicates).
The code has the following form:

$$\tilde{n}\#\tilde{R_1^M}\#\ldots\#\tilde{R_n^M}, \text{ where:}$$

- $\tilde{n}$ is the part coding the universe of the model and consists of n 1s.
- $\tilde{R_i^M}$ — the code for the n_i-ary relation R_i^M — is an n^{n_i}-bit string whose j-th bit is 1 iff the j-th tuple in U^{n_i} (ordered lexicographically) is in R_i^M.
- # is a separating symbol.

Coding Example

Consider vocabulary $\sigma = \{P, R\}$, where P is a unary predicate and R a binary relation. Take the σ-model $\mathbb{M} = (M, P^M, R^M)$, where the universe $M = \{1, 2, 3\}$, the unary relation $P^M \subseteq M$ is equal to $\{2\}$ and the binary relation $R^M \subseteq M^2$ consists of the pairs $(2, 2)$ and $(3, 2)$.

- $\tilde{n}$ consists of three 1s as there are three elements in M.
- $\tilde{P^M}$ is the string of length three with 1s in places corresponding to the elements from M belonging to P^M. Hence $\tilde{P^M} = 010$ as $P^M = \{2\}$.
- $\tilde{R^M}$ is obtained by writing down all $3^2 = 9$ binary strings of elements from M in lexicographical order and substituting 1 in places corresponding to the pairs belonging to R^M and 0 in all other places. As a result $\tilde{R^M} = 000010010$.

Adding all together the code for $\mathbb{M}$ is $111\#010\#000010010$.

Iteration

1. Most logicians criticized some papers.
2. It(most, some)[Logicians, Papers, Criticized].

Definition

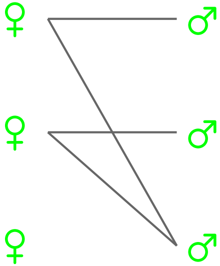
Let Q and Q′ be generalized quantifiers of type (1, 1). Let A, B be subsets of the universe and R a binary relation over the universe. Suppressing the universe, we will define the *iteration* operator as follows:

$$\mathrm{It}(\mathrm{Q}, \mathrm{Q}')[A, B, R] \iff \mathrm{Q}[A, \{a \mid \mathrm{Q}'[B, R_{(a)}]\}],$$

where $R_{(a)} = \{b \mid R(a, b)\}$.

Illustration

- Most girls and most boys hate each other.



Iteration is easy

Theorem (Steinert-Threlkeld & Icard)

Let Q *and* Q' *be computable by DFA (PDA), then* $\mathrm{It}(\mathrm{Q}, \mathrm{Q}')$ *is also DFA (PDA) computable.*

Cumulation

- Eighty professors taught sixty courses at ESSLLI'08.

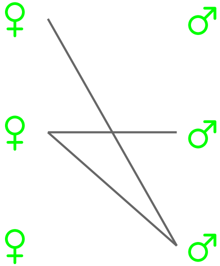
Definition

$\mathrm{Cum}(\mathrm{Q}, \mathrm{Q}')[A, B, R] \iff$

$$\mathrm{It}(\mathrm{Q}, \mathrm{some})[A, B, R] \wedge \mathrm{It}(\mathrm{Q}', \mathrm{some})[B, A, R^{-1}]$$

Illustration

- Most girls and most boys hate each other.



Possibly branching sentences

1. Most villagers and most townsmen hate each other.
2. One third of villagers and half of townsmen hate each other.
3. 5 villagers and 7 townsmen hate each other.

Branching reading

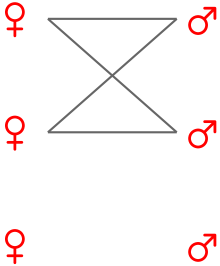
- Most girls and most boys hate each other.

$$\begin{matrix} \text{most } x : G(x) \\ \text{most } y : B(y) \end{matrix} \; H(x, y).$$

$$\exists A \exists A' [\text{most}(G, A) \wedge \text{most}(B, A') \wedge \forall x \in A \, \forall y \in A' \; H(x, y)].$$

Illustration

- Most girls and most boys hate each other.



Definition

Definition

Let Q and Q′ be both MON↑ quantifiers of type $(1,1)$. Define the *branching* of quantifier symbols Q and Q′ as the type (1, 1, 2) quantifier symbol $\mathrm{Br}(\mathrm{Q},\mathrm{Q}')$. A structure $\mathbb{M} = (M, A, B, R) \in \mathrm{Br}(\mathrm{Q},\mathrm{Q}')$ if the following holds:

$$\exists X \subseteq A\, \exists Y \subseteq B[(X,A) \in \mathrm{Q} \wedge\ (Y,B) \in \mathrm{Q}' \wedge\ X \times Y \subseteq R].$$

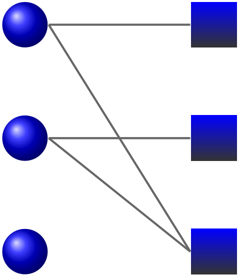
Branching readings are intractable

Theorem

Proportional branching sentences are NP-complete.

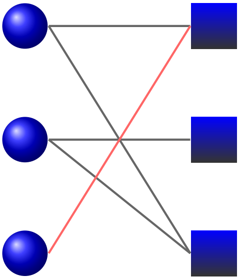
Two-way quantification

$$\mathrm{It}(Q_1, Q_2) \wedge \mathrm{It}(Q_2, Q_1)$$



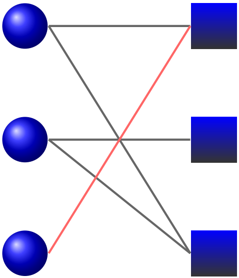
Two-way quantification

$$\mathrm{It}(Q_1, Q_2) \wedge \mathrm{It}(Q_2, Q_1)$$



Two-way quantification

$$\mathrm{It}(Q_1, Q_2) \wedge \mathrm{It}(Q_2, Q_1)$$



Subjects are happy to accept such interpretation.

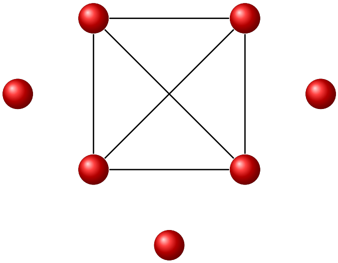
Gierasimczuk & Szymanik, Branching Quantification vs. Two-way Quantification, Journal of Semantics, 2009

Potentially strong reciprocal sentences

1. Andi, Jarmo and Jakub laughed at one another.
2. 15 men are hitting one another.
3. Most of the PMs refer to each other.

Strong reading

- Most of the PMs refer to each other.



Strong reciprocal lift

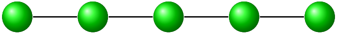
Definition

Let Q be a right monotone increasing quantifier of type (1, 1). We define:

$$\mathrm{Ram}_{\mathrm{S}}(\mathrm{Q})[A, R] \iff \exists X \subseteq A[\mathrm{Q}(A, X) \\ \wedge \forall x, y \in X(x \neq y \implies R(x, y))].$$

Intermediate reading

- Most Boston pitchers sat alongside each other.



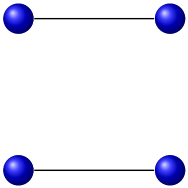
Intermediate reciprocal lift

Definition

$$\mathrm{Ram_I}(\mathrm{Q})[A, R] \iff \exists X \subseteq A[\mathrm{Q}(A, X)$$
$$\wedge \forall x, y \in X(x \neq y \implies \exists \text{ sequence } z_1, \ldots, z_\ell \in X \text{ such that}$$
$$(z_1 = x \wedge R(z_1, z_2) \wedge \ldots \wedge R(z_{\ell-1}, z_\ell) \wedge z_\ell = y)].$$

Weak reading

- Some pirates were staring at each other in surprise.



Weak reciprocal lift

Definition

$$\mathrm{Ram}_{\mathrm{W}}(\mathrm{Q})[A, R] \iff \exists X \subseteq A[\mathrm{Q}(A, X) \\ \wedge \forall x \in X\, \exists y \in X(x \neq y \wedge R(x, y))].$$

Strong Meaning Hypothesis

Hypothesis

Reading associated with the reciprocal in a given sentence is the strongest available reading which is consistent with relevant information supplied by the context.

Strong Meaning Hypothesis

Hypothesis

Reading associated with the reciprocal in a given sentence is the strongest available reading which is consistent with relevant information supplied by the context.

Example

1. The children followed each other into the church.
2. The children followed each other around the Maypole.

Dalrymple et al., Reciprocal Expressions and the Concept of Reciprocity. Linguistics and Philosophy, 1998.

Szymanik, Computational complexity of polyadic lifts of generalized quantifiers in natural language. L&P 2010.

Research question

Draw:

1. All/Most of the dots are connected to each other.

Research question

Draw:

1. All/Most of the dots are connected to each other.

- Against SMH:
  - Ambiguous between strong and intermediate.

Research question

Draw:

1. All/Most of the dots are connected to each other.

- Against SMH:
  - Ambiguous between strong and intermediate.
- In line with complexity: fewer strong pictures for ‘most’.

Bott et al., Interpreting Tractable versus Intractable Reciprocal Sentences, Proceedings of the International Conference on Computational Semantics, 2011.

Schlotterbeck & Bott, Easy solutions for a hard problem? The computational complexity of reciprocals with quantificational antecedents, Proc. of the Logic & Cognition Workshop at ESSLLI 2012.

Outline

Generalized Quantifiers

Semantic universale

Monotonicity patterns

GQs in logic

Languages and automata

Computing quantifiers

Computational Complexity

Complex GQs
- Polyadic quantifiers
  - Branching Quantifiers
  - Strong Reciprocity
- Collective quantifiers

Collectivity

(1.) All the Knights but King Arthur *met in secret*.

(2.) Most climbers *are friends*.

(3.) John and Mary *love each other*.

(4.) The samurai *were twelve in number*.

(5.) Many girls *gathered*.

(6.) Soldiers *surrounded* the Alamo.

(7.) Tikitu and Samson *lifted* the table.

Let’s start with examples

(1.) Five people lifted the table.

Let's start with examples

(1.) Five people lifted the table.

(1'.) $\exists^{=5}x[\text{People}(x) \wedge \text{Lift}(x)]$.

Let's start with examples

(1.) Five people lifted the table.

(1'.) $\exists^{=5} x[\text{People}(x) \wedge \text{Lift}(x)]$.

(1''.) $\exists X[X \subseteq \text{People} \wedge \text{Card}(X) = 5 \wedge \text{Lift}(X)]$.

Let's start with examples

(1.) Five people lifted the table.

(1'.) $\exists^{=5} x[\text{People}(x) \wedge \text{Lift}(x)]$.

(1''.) $\exists X[X \subseteq \text{People} \wedge \text{Card}(X) = 5 \wedge \text{Lift}(X)]$.

(2.) Some students played poker together.

Let's start with examples

(1.) Five people lifted the table.

(1'.) $\exists^{=5} x[\text{People}(x) \wedge \text{Lift}(x)]$.

(1''.) $\exists X[X \subseteq \text{People} \wedge \text{Card}(X) = 5 \wedge \text{Lift}(X)]$.

(2.) Some students played poker together.

(2'.) $\exists X[X \subseteq \text{Students} \wedge \text{Play}(X)]$.

Existential modifier

Definition (van der Does 1992)

Fix a universe of discourse U and take any $X \subseteq U$ and $Y \subseteq \mathcal{P}(U)$. Define the existential lift Q^{EM} of a quantifier Q in the following way:

$$\mathrm{Q}^{EM}(X, Y) \text{ is true } \iff \exists Z \subseteq X[\mathrm{Q}(X, Z) \wedge Z \in Y].$$

Existential modifier

Definition (van der Does 1992)

Fix a universe of discourse U and take any $X \subseteq U$ and $Y \subseteq \mathcal{P}(U)$. Define the existential lift Q^{EM} of a quantifier Q in the following way:

$$\mathrm{Q}^{EM}(X, Y) \text{ is true } \iff \exists Z \subseteq X[\mathrm{Q}(X, Z) \wedge Z \in Y].$$

$$((et)((et)t)) \rightsquigarrow ((et)(((et)t)t))$$

Van Benthem problem

Observation

$(\cdot)^{EM}$ works only for right monotone increasing quantifiers.

Van Benthem problem

Observation

$(\cdot)^{EM}$ *works only for right monotone increasing quantifiers.*

(1.) No students met yesterday at the coffee shop.

Van Benthem problem

Observation

$(\cdot)^{EM}$ *works only for right monotone increasing quantifiers.*

(1.) No students met yesterday at the coffee shop.

— $\downarrow$MON$\downarrow \leadsto \uparrow$MON$\uparrow$

Van Benthem problem

Observation
$(\cdot)^{EM}$ *works only for right monotone increasing quantifiers.*

(1.) No students met yesterday at the coffee shop.

— $\downarrow$MON$\downarrow \rightsquigarrow \uparrow$MON$\uparrow$

(2.) No left-wing students met yesterday at the coffee shop.

(3.) No students met yesterday at the “Che” coffee shop.

The total number is missing

(1.) Exactly 5 students drank a whole keg of beer together.

The total number is missing

(1.) Exactly 5 students drank a whole keg of beer together.

(1'.) $(\exists^{=5})^{EM}$[Student, Drink-a-whole-keg-of-beer].

The total number is missing

(1.) Exactly 5 students drank a whole keg of beer together.

(1'.) $(\exists^{=5})^{EM}$[Student, Drink-a-whole-keg-of-beer].

(1''.) $\exists A \subseteq \text{Student}[\text{card}(A) = 5 \wedge \text{Drink-a-whole-keg-of-beer}(A)]$

Neutral Modifier

Definition (van der Does 1992)

Let U be a universe, $X \subseteq U$, $Y \subseteq \mathcal{P}(U)$, and Q a type (1, 1) quantifier. We define the *neutral modifier*:

$$\mathrm{Q}^N[X, Y] \text{ is true } \iff \mathrm{Q}\left[X, \bigcup(Y \cap \mathcal{P}(X))\right].$$

Monotonicity preservation under $(\cdot)^N$

Fact (Ben-Avi and Winter 2003)

Let Q *be a distributive determiner. If* Q *belongs to one of the classes* $\uparrow$*MON*$\uparrow$*,* $\downarrow$*MON*$\downarrow$*, MON*$\uparrow$*, MON*$\downarrow$*, then the collective determiner* Q^N *belongs to the same class. Moreover, if* Q *is conservative and* $\sim$*MON (MON*$\sim$*), then* Q^N *is also* $\sim$*MON (MON*$\sim$*).*

What about split groups?

(1.) Exactly 5 students drank a whole keg of beer together.

What about split groups?

(1.) Exactly 5 students drank a whole keg of beer together.

(1'.) $(\exists^{=5})^N$[Student, Drink-a-whole-keg-of-beer].

What about split groups?

(1.) Exactly 5 students drank a whole keg of beer together.

(1'.) $(\exists^{=5})^{N}$[Student, Drink-a-whole-keg-of-beer].

$$\text{card}\big(\{x | \exists A \subseteq \text{Student}[x \in A \wedge \text{Drink-a-whole-keg-of-beer}(A)]\}\big) = 5.$$

Determiner fitting

Definition (Winter 2001)

For all $X, Y \subseteq \mathcal{P}(U)$ we have that

$$\mathrm{Q}^{\mathrm{dfit}}(X, Y) \text{ is true}$$

$$\Longleftrightarrow$$

$$\mathrm{Q}[\cup X, \cup(X \cap Y)] \wedge [X \cap Y = \emptyset \vee \exists W \in X \cap Y \wedge \mathrm{Q}(\cup X, W)].$$

Determiner fitting

Definition (Winter 2001)

For all $X, Y \subseteq \mathcal{P}(U)$ we have that

$$\mathrm{Q}^{\mathrm{dfit}}(X, Y) \text{ is true}$$

$$\Longleftrightarrow$$

$$\mathrm{Q}[\cup X, \cup(X \cap Y)] \wedge [X \cap Y = \emptyset \vee \exists W \in X \cap Y \wedge \mathrm{Q}(\cup X, W)].$$

$$((et)((et)t)) \rightsquigarrow (((et)t)(((et)t)t))$$

Dfit works

(1.) Exactly 5 students drank a whole keg of beer together.

Dfit works

(1.) Exactly 5 students drank a whole keg of beer together.

(1'.) $(\exists^{=5})^{\text{dfit}}[\text{Student}, \text{Drink-a-whole-keg-of-beer}]$.

Dfit works

(1.) Exactly 5 students drank a whole keg of beer together.

(1'.) $(\exists^{=5})^{\text{dfit}}[\text{Student}, \text{Drink-a-whole-keg-of-beer}]$.

$$\text{card}\big(\{x \in A | A \subseteq \text{Student} \wedge \text{Drink-a-whole-keg-of-beer}(A)\}\big) = 5$$
$$\wedge \exists W \subseteq \text{Student}[\text{Drink-a-whole-keg-of-beer}(W) \wedge \text{card}(W) = 5].$$

It really works...

Monotonicity of Q	Monotonicity of Q^{dfit}	Example
$\uparrow$MON$\uparrow$	$\uparrow$MON$\uparrow$	Some
$\downarrow$MON$\downarrow$	$\downarrow$MON$\downarrow$	Less than five
$\downarrow$MON$\uparrow$	$\sim$MON$\uparrow$	All
$\uparrow$MON$\downarrow$	$\sim$MON$\downarrow$	Not all
$\sim$MON$\sim$	$\sim$MON$\sim$	Exactly five
$\sim$MON$\downarrow$	$\sim$MON$\downarrow$	Not all and less than five
$\sim$MON$\uparrow$	$\sim$MON$\uparrow$	Most
$\downarrow$MON$\sim$	$\sim$MON$\sim$	All or less than five
$\uparrow$MON$\sim$	$\sim$MON$\sim$	Some but not all

Table : Monotonicity under the determiner fitting operator; cf. (Ben-Avi and Winter 2003).

...but violates invariance properties

Definition

A distributive determiner of type (1, 1) is conservative if and only if the following holds for all M and all $A, B \subseteq M$:

$$\mathrm{Q}_M[A, B] \iff Q_M[A, A \cap B].$$

...but violates invariance properties

Definition

A distributive determiner of type (1, 1) is conservative if and only if the following holds for all M and all $A, B \subseteq M$:

$$\mathrm{Q}_M[A, B] \iff Q_M[A, A \cap B].$$

Fact

For every Q *the quantifiers* Q^{EM}, Q^{N}, *and* Q^{dfit} *are not CONS.*

...and not only because of technicalities

Definition

We say that a collective determiner Q of type $((et)(((et)t)t))$ satisfies *collective conservativity* iff the following holds for all M and all $A, B \subseteq M$:

$$\mathrm{Q}_M[A, B] \iff Q_M[A, \mathcal{P}(A) \cap B].$$

...and not only because of technicalities

Definition

We say that a collective determiner Q of type $((et)(((et)t)t))$ satisfies *collective conservativity* iff the following holds for all M and all $A, B \subseteq M$:

$$\mathrm{Q}_M[A, B] \iff Q_M[A, \mathcal{P}(A) \cap B].$$

Fact

For every Q *the collective quantifiers* Q^{EM}, Q^{N}, *and* Q^{dfit} *satisfy collective conservativity.*

Invariance properties are forced

- Conservativity incorporated into the lifts.
- We need less arbitrary approach.

Second-order GQs

$$
\begin{aligned}
\exists^2 &= \{(M,P) \mid P \subseteq \mathcal{P}(M) \ \&\ P \neq \emptyset\}. \\
\textsc{Even} &= \{(M,P) \mid P \subseteq \mathcal{P}(M) \ \&\ \mathrm{card}(P) \text{ is even}\}. \\
\textsc{Even}' &= \{(M,P) \mid P \subseteq \mathcal{P}(M) \ \&\ \forall X \in P(\mathrm{card}(X) \text{ is even})\}. \\
\textsc{Most} &= \{(M,P,S) \mid P,S \subseteq \mathcal{P}(M) \ \&\ \mathrm{card}(P \cap S) > \mathrm{card}(P - S)\}.
\end{aligned}
$$

Second-order GQs

$$
\begin{aligned}
\exists^2 &= \{(M, P) \mid P \subseteq \mathcal{P}(M) \;\&\; P \neq \emptyset\}. \\
\textsc{Even} &= \{(M, P) \mid P \subseteq \mathcal{P}(M) \;\&\; \mathsf{card}(P) \text{ is even}\}. \\
\textsc{Even}' &= \{(M, P) \mid P \subseteq \mathcal{P}(M) \;\&\; \forall X \in P(\mathsf{card}(X) \text{ is even})\}. \\
\textsc{Most} &= \{(M, P, S) \mid P, S \subseteq \mathcal{P}(M) \;\&\; \mathsf{card}(P \cap S) > \mathsf{card}(P - S)\}.
\end{aligned}
$$

Observation

SOGQs do not decide invariance properties!

Second-order GQs

$$
\begin{array}{rcl}
\exists^2 & = & \{(M,P) \mid P \subseteq \mathcal{P}(M) \;\&\; P \neq \emptyset\}. \\
\textsc{Even} & = & \{(M,P) \mid P \subseteq \mathcal{P}(M) \;\&\; \mathrm{card}(P) \text{ is even}\}. \\
\textsc{Even}' & = & \{(M,P) \mid P \subseteq \mathcal{P}(M) \;\&\; \forall X \in P(\mathrm{card}(X) \text{ is even})\}. \\
\textsc{Most} & = & \{(M,P,S) \mid P,S \subseteq \mathcal{P}(M) \;\&\; \mathrm{card}(P \cap S) > \mathrm{card}(P - S)\}.
\end{array}
$$

Observation

SOGQs do not decide invariance properties!

Question

How invariance properties interact with definability?

Warning!

Do not confuse:

Warning!

Do not confuse:

- FO GQs (Lindström) with FO-definable quantifiers

  E.g. most is FO GQs but is not FO-definable.

Warning!

Do not confuse:

- FO GQs (Lindström) with FO-definable quantifiers

  E.g. most is FO GQs but is not FO-definable.
- SO GQs with SO-definable quantifiers

  E.g. MOST is SO GQs but probably not SO-definable.

GQs are not enough

Theorem (Kontinen 2002)

The extension $\mathcal{L}^*$ of first-order logic by all Lindström quantifiers cannot define the monadic second-order existential quantifier.

GQs are not enough

Theorem (Kontinen 2002)

The extension $\mathcal{L}^*$ of first-order logic by all Lindström quantifiers cannot define the monadic second-order existential quantifier.

Corollary

Lindström quantifiers alone are not adequate for formalizing all natural language quantification.

GQs are not enough

Theorem (Kontinen 2002)

The extension $\mathcal{L}^*$ of first-order logic by all Lindström quantifiers cannot define the monadic second-order existential quantifier.

Corollary

Lindström quantifiers alone are not adequate for formalizing all natural language quantification.

Example

Some students gathered to play poker.

For example . . .

Definition

Denote by some^{EM}:

$$\{(M, P, G) \mid P \subseteq M;\ G \subseteq \mathcal{P}(M) :\ \exists Y \subseteq P(Y \neq \emptyset \ \&\ P \in G)\}.$$

For example . . .

Definition

Denote by some^{EM}:

$$\{(M, P, G) \mid P \subseteq M;\ G \subseteq \mathcal{P}(M) :\ \exists Y \subseteq P(Y \neq \emptyset \ \&\ P \in G)\}.$$

(3.) Some students played poker together.

For example . . .

Definition

Denote by some^{EM}:

$$\{(M, P, G) \mid P \subseteq M;\; G \subseteq \mathcal{P}(M) : \; \exists Y \subseteq P(Y \neq \emptyset \;\&\; P \in G)\}.$$

(3.) Some students played poker together.

(3'.) some^{EM} $x, X[\text{Student}(x), \text{Play}(X)]$.

Another example . . .

Definition

We take five^{EM} to be the second-order quantifier denoting:

$$\{(M, P, G) \mid P \subseteq M;\ G \subseteq \mathcal{P}(M) :\ \exists Y \subseteq P(\mathrm{card}(Y) = 5\ \&\ P \in G)\}.$$

Another example . . .

Definition

We take five^{EM} to be the second-order quantifier denoting:

$$\{(M, P, G) \mid P \subseteq M;\ G \subseteq \mathcal{P}(M) :\ \exists Y \subseteq P(\text{card}(Y) = 5 \ \& \ P \in G)\}.$$

(4.) Five people lifted the table.

Another example . . .

Definition

We take five^{EM} to be the second-order quantifier denoting:

$$\{(M, P, G) \mid P \subseteq M;\ G \subseteq \mathcal{P}(M) :\ \exists Y \subseteq P(\text{card}(Y) = 5\ \&\ P \in G)\}.$$

(4.) Five people lifted the table.

(4'.) $\text{five}^{EM} x, X[\text{Student}(x), \text{Lift}(X)]$.

SO-definable GQs are closed on lifts

Theorem

Let Q *be a Lindström quantifier definable in* SO*. Then the collective quantifiers* Q^{EM}, Q^{N}, *and* $\mathsf{Q}^{\mathrm{dfit}}$ *are definable in* SO*.*

SO-definable GQs are closed on lifts

Theorem

Let Q *be a Lindström quantifier definable in* SO*. Then the collective quantifiers* Q^{EM}*,* Q^{N}*, and* $\mathsf{Q}^{\mathrm{dfit}}$ *are definable in* SO*.*

Proof.

Let us consider the case of Q^{EM}. Let $\psi(x)$ and $\phi(Y)$ be formulas. We want to express $\mathsf{Q}^{EM}x, Y(\psi(x), \phi(Y))$ in second-order logic. By the assumption, the quantifier Q can be defined by some sentence $\theta \in \mathrm{SO}[\{P_1, P_2\}]$. We can now use the following formula:

$$\exists Z(\forall x(Z(x) \to \psi(x)) \wedge (\theta(P_1/\psi(x), P_2/Z) \wedge \phi(Y/Z)).$$

□

And this is the case for all SO-definable lifts

Theorem

Let us assume that the lift $(\cdot)^*$ *and a Lindström quantifier* Q *are both definable in second-order logic. Then the collective quantifier* Q^* *is also definable in second-order logic.*

Some collectives are not definable in SO

Theorem (Kontinen and Szymanik 2012)

The quantifier MOST *is not definable in second-order logic.*

Proof.

By translating into FO$(+, \times)$ over cardinalities 2^n and using Ajtai's 1983 results. $\square$

Consequences

Corollary

The type-shifting strategy is not general enough to cover all collective quantification in natural language.

What is the right ontology for semantics?

- $\mathcal{L}^*$ and SO doesn't capture natural language?

What is the right ontology for semantics?

- $\mathcal{L}^*$ and SO doesn't capture natural language?
- Are many-sorted (algebraic) models more plausible?
  - Type-shifting is too complex;
  - In principle this question is psychologically testable.

Σ_1^1(Ristad's)-thesis

Hypothesis

Our everyday language is semantically bounded by the properties expressible in the existential fragment of second-order logic.

- Does SOGQ “MOST” belong to everyday language?
  - Everyday language doesn’t realize prop. coll. qua.
  - No need to extend the higher-order approach to prop. qua.

- Does SOGQ “MOST” belong to everyday language?
  - Everyday language doesn’t realize prop. coll. qua.
  - No need to extend the higher-order approach to prop. qua.

Question

Have we just encountered an example where complexity restricts the expressibility of everyday language as suggested by Σ_1^1*-thesis?*